TWO APPLICATIONS OF TSCHIRNHAUS TRANSFORMATIONS IN THE ELEMENTARY THEORY OF EQUATIONS

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In a beginning course in the theory of equations the student is introduced to a number of very simple transformations of equations, such as $y=mx$, $y=x-a$, $y=1/x$, and perhaps a few others. In his later work, if he continues the study of algebraic equations, he meets transformations of much more complicated nature in the works of Brioschi, Cayley, Hermite, Gordan, Weber, and others, in connection with quintic normal forms, invariants, and various topics. It seems to me both possible and desirable to lead up to these more difficult transformations by making some use, in the elementary course, of Tschirnhaus transformations of higher than the first degree.[1] This step is also justified by the fact that such transformations can be employed advantageously in treating some of the usual matters. Further, they serve as an application of symmetric functions, especially those usually denoted by s_k, where s_k means the sum of the kth powers of the roots of a given equation.

In this paper I wish to point out two important and familiar topics which can be discussed effectively with the aid of quadratic transformations, namely, the evaluation of the discriminant of the general cubic equation in terms of the coefficients, and the development of criteria for the number of real roots of the general quartic equation. Each of these can be handled in a number of different ways, as the reader doubtless knows, but the present method may nevertheless be of some interest and value.

To obtain the value of the discriminant of the general cubic, taken for convenience in the reduced form[2]

$$x^3 + px + q = 0, \tag{1}$$

with roots x_1, x_2, x_3, we apply to (1) the transformation $y=x^2$. The transformed equation can be set up in a number of ways. If its roots are denoted by $y_1=x_1^2$, $y_2=x_2^2$, $y_3=x_3^2$, their sum, the sum of their products taken two at a time, and their product can be easily computed with the aid of symmetric functions of (1). [We shall assume throughout this presentation that the symmetric functions which are required have been studied in an earlier part of the course.] Or we may follow the procedure customary in Tschirnhaus transformations and find the coefficients of the transformed equation from $\sum y$, $\sum y^2$ and $\sum y^3$, which can also be determined with the aid of $y=x^2$ and (1). (The symbol $\sum y$ means $y_1+y_2+y_3$, $\sum y^2$ means $y_1^2+y_2^2+y_3^2$, and so on.) In the present case this method is a little longer than the other. Or, we may use a special device, which consists in writing (1) as $x(x^2+p)=-q$, replacing x^2 by y, squaring and again replacing x^2 by y.

[1] This suggestion is, of course, not new, being carried out in some texts and, undoubtedly, by many teachers. It might well be extended to cover linear fractional transformations as well.

[2] The coefficients in (1) and (3) are real.

The transformed equation is found to be

$$F(y) = y(y + p)^2 - q^2 = 0, \tag{2}$$

or

$$y^3 + 2py^2 + p^2y - q^2 = 0.$$

Its factored form is

$$(y - x_1^2)(y - x_2^2)(y - x_3^2) = 0.$$

Now by definition the discriminant of (1) is

$$(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2,$$

and it will be denoted by D. Since

$$x_1 + x_2 + x_3 = 0, \quad x_1x_2 + x_1x_3 + x_2x_3 = p, \quad x_1^2 + x_2^2 + x_3^2 = -2p,$$

we find

$$x_1x_2 = x_3^2 + p, \quad (x_1 - x_2)^2 = -3x_3^2 - 4p = 3\left(-\frac{4p}{3} - x_3^2\right).$$

The other factors of D reduce similarly, so that

$$D = 27\left(-\frac{4p}{3} - x_1^2\right)\left(-\frac{4p}{3} - x_2^2\right)\left(-\frac{4p}{3} - x_3^2\right).$$

But this is 27 multiplied by $F(-4p/3)$, which gives at once $D = -4p^3 - 27q^2$. I know no more convenient method of securing this result.

Equation (2) can also be used in the investigation of the relation between the value of D and the number of real roots of (1), since, from $y = x^2$, there is a one-to-one correspondence between the real roots of (1) and the *positive* real roots of (2). To discuss the latter, we may consider the graph of $F(y) = y^3 + 2py^2 + p^2y - q^2$. The derivative $F'(y)$ vanishes at $y = -p/3$ and $y = -p$, and $F(0) = -q^2$, $F(-p/3) = D/27$, $F(-p) = -q^2$. If p is positive, and D therefore negative, $F(y)$ obviously increases continuously as y increases from zero; since $F(0)$ is negative[1] $F(y) = 0$ has one and only one positive root. If p is negative and D also negative, the maximum and minimum points on the graph are both to the right of $y = 0$ and below the y-axis; again there can be only one positive root. If p is negative and D is zero, $F(y) = 0$ has a double root, since the maximum point of the graph is at $y = -p/3$, $F(y) = 0$. The third root is also positive. If p is negative and D is positive, the maximum point of the graph is *above* the y-axis, while the minimum point is below; hence there must be three positive roots. The student is here supposed to be familiar with the essential features of the graph of a cubic polynomial.

We pass now to the consideration of the general reduced quartic,

[1] The trivial case $q = 0$ may be treated separately.

$$x^4 + qx^2 + rx + s = 0, \tag{3}$$

with roots x_1, x_2, x_3, x_4. Its discriminant is, by definition the product

$$(x_1 - x_2)^2(x_1 - x_3)^2(x_1 - x_4)^2(x_2 - x_3)^2(x_2 - x_4)^2(x_3 - x_4)^2,$$

which we shall again call D. The value of D in terms of q, r, s is easily obtainable, since it is equal, except possibly for a constant factor, to the discriminant of any of the resolvent cubics which are commonly introduced in the solution of (3). This is proved in almost any text on the subject, and gives a simple and satisfactory evaluation of D. It is, however, not quite so easy to determine conditions on q, r, s, D which will insure that (3) has a certain definite number of real roots. Some authors obtain these conditions with the aid of Sturm's functions, while others use the resolvent cubic; we shall employ a quadratic transformation and a few symmetric functions.

In the first place, it is well known that D will be positive if (3) has four real roots or two pairs of conjugate complex roots, while D will be negative if (3) has two real roots and two complex. These statements can be proved at once, by choosing the notation for the roots so as to exhibit their reality or complexity and then substituting in the above value of D. Thus, to treat the case of four complex roots, we may write

$$x_1 = a + bi, \quad x_2 = a - bi, \quad x_3 = -a + ci, \quad x_4 = -a - ci,$$

after which it is easy to see that the product $(x_1 - x_3)(x_2 - x_3)$ is positive, the product $(x_1 - x_4)(x_2 - x_4)$ is positive, while $(x_1 - x_2)(x_3 - x_4) = -4bc$. Therefore, D is surely positive. Conversely, we may say that if D is negative (3) will have two (distinct) real roots and two complex, while if D is positive (3) will have all its roots real or none of them real. That is, the character of D alone is no longer sufficient, as it is for the cubic, to determine completely the number of real roots. However, if D is positive, the following familiar conditions can be stated:

$$\begin{aligned} &\text{If } D > 0,\ q \geqq 0,\ (3) \text{ has no real roots,} \\ &\text{If } D > 0,\ L \leqq 0,\ (3) \text{ has no real roots, } (L \equiv 8qs - 2q^3 - 9r^2), \\ &\text{If } D > 0,\ q < 0,\ L > 0,\ (3) \text{ has four real roots.} \end{aligned} \tag{4}$$

The first of these follows at once, since, for (3), $s_2 = -2q$, and s_2 cannot be zero or negative if all the roots are real and distinct. To prove the third, we note that, if $q < 0$ and $L > 0$, $q^2 - 4s$ must be positive. The conditions $q < 0$, $q^2 - 4s > 0$ are equivalent to $s_2 > 0$, $s_4 - \frac{1}{4} s_2^2 > 0$. Now suppose that (3) has no real roots, so that

$$x_1 = a + bi, \quad x_2 = a - bi, \quad x_3 = -a + ci, \quad x_4 = -a - ci,$$

with b and c different from zero. Then, by simple algebra,

$$s_2 = 4a^2 - 2b^2 - 2c^2, \quad s_4 - \tfrac{1}{4}s_2^2 = (b^2 - c^2)^2 - 8a^2(b^2 + c^2).$$

Now if $s_2 > 0$, $8a^2 > 4b^2 + 4c^2$, and $s_4 - \frac{1}{4}s_2^2 > 0$ implies

$$(b^2 - c^2)^2 > 4(b^2 + c^2)^2,$$

which is clearly false. Hence (3) can not have no real roots, and must have four.[1]

The second condition of (4) does not seem to lend itself to similar treatment; we now proceed to apply a transformation to (3), so chosen that the present case will reduce to the first case of (4). That is, we make use of

$$y = x^2 + k_1 x + k_2, \tag{5}$$

where the parameters k_1 and k_2 are to be determined so that $\sum y = \sum y^2 = 0$, that is, so that the transformed equation in y will have no term in y^3 or y^2. Now $\sum y = s_2 + k_1 s_1 + 4k_2$, which is zero if $k_2 = q/2$. Now square (5) and sum over the roots of (3); the equation $\sum y^2 = 0$ turns out to be a quadratic[2] in k_1 whose discriminant is exactly $-4L$, as defined above. Hence, if L is not positive, k_1 is real, and we are thus able, by a real transformation, to reduce (3) to an equation in y which has no terms in y^3 or y^2 and cannot, therefore, have all its roots real. But (5) shows that, when k_1 and k_2 are real, each real root of (3) will give a real root of the transformed equation. We conclude that (3) cannot have four real roots, and since $D > 0$, it must have no real roots.

Here then are two fundamental matters in the theory of equations which can be treated conveniently with the aid of simple transformations. In each case the desired results are obtained, I believe, as easily as is possible, and, in addition, the student is becoming acquainted with an algebraic tool of some importance.

H. von KOCH'S FIRST LEMMA AND ITS GENERALIZATION

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In a letter to Poincaré, Helge von Koch wrote the following in 1895:[3]

LEMMA 1. *$\alpha_1, \alpha_2, \cdots$, and $\beta_1, \beta_2, \cdots$, being any given quantities whatever, the necessary and sufficient condition for the absolute convergence of the infinite determinant*

$$\begin{vmatrix} 1 & \alpha_1 & 0 & 0 & \cdots \\ \beta_1 & 1 & \alpha_2 & 0 & \cdots \\ 0 & \beta_2 & 1 & \alpha_3 & \cdots \\ 0 & 0 & \beta_3 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

is that the series

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[1] No claim of originality is made for this argument, though I have not seen it used elsewhere.

[2] $\sum y^2 = -2qk_1^2 - 6rk_1 + (q^2 - 4s)$. The reader may take $k_1 = 0$ and show that (4) can be written with $q^2 - 4s$ in place of L.

[3] See Comptes Rendus de l'Academie des Sciences, vol. 120 (1895), p. 144. See also Whittaker and Watson, *Modern Analysis*, 3d edition, p. 37.

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \cdots$$

shall be absolutely convergent.

It is the purpose of this paper to give a proof of the above lemma, and to show, in addition, that the convergence of the given determinant depends on the absolute convergence of the series

$$\alpha_1 + \beta_1 + \alpha_2 + \beta_2 + \alpha_3 + \beta_3 + \cdots .$$

As a generalization of von Koch's lemma, we shall give also proofs of convergence of more general infinite determinants which arise in the solution of homogeneous linear difference equations by means of infinite determinants.[1]

1. *Proof*: Consider this determinant of order m:

$$D_m = \begin{vmatrix} 1 & \alpha_1 & 0 & 0 & \cdots & 0 \\ \beta_1 & 1 & \alpha_2 & 0 & \cdots & 0 \\ 0 & \beta_2 & 1 & \alpha_3 & \cdots & 0 \\ 0 & 0 & \beta_3 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots \beta_m & 1 \end{vmatrix};$$

and, to compare with it, construct the auxiliary determinant, Δ_m, all the elements in which are zeros except those of the principal diagonal, which are $1+|\alpha_m\beta_m|$, $m=1, 2, \cdots, m$:

$$\Delta_m = \begin{vmatrix} 1+|\alpha_1\beta_1| & 0 & 0 & \cdots & 0 \\ 0 & 1+|\alpha_2\beta_2| & 0 & \cdots & 0 \\ 0 & 0 & 1+|\alpha_3\beta_3| & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1+|\alpha_m\beta_m| \end{vmatrix}.$$

Now in the expansion of Δ_m all the terms are positive and are formed by certain[2] combinations of $2, 4, 6, \cdots, 2m$ letters at a time, except the first term which is unity. Hence Δ_m contains the absolute values of all the terms of D_m plus additional terms all positive, since the upper limit of the suffixes in the development of Δ_m is m while that of D_m is $m-1$; and since some of the terms of D_m are negative, it follows at once that Δ_m is greater than $|D_m|$.

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[1] See the author's thesis (the solution of homogeneous linear difference equations by means of infinite determinants) which is in the library of the University of California, Berkeley, California.

[2] But not every possible combination; for when $m=3$, $\Delta_3 = 1+\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3+\alpha_1\alpha_2\beta_1\beta_2 + \alpha_1\alpha_3\beta_1\beta_3+\alpha_2\alpha_3\beta_2\beta_3+\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3$ which does *not* contain such terms as $\alpha_1\beta_2$, $\alpha_1\alpha_3\alpha_2\beta_1$, etc.

Similarly the difference $(\Delta_{m+p}-\Delta_m)$ contains the absolute values of all the terms of the difference $(D_{m+p}-D_m)$ and other terms all positive. Consequently

$$|D_{m+p}-D_m| < \Delta_{m+p}-\Delta_m.$$

Hence the sequence D_m converges if the sequence Δ_m converges.

Now Δ_m is convergent if the infinite product of the terms of the main diagonal is convergent, since Δ_m and that infinite product are identical.[1] But that infinite product is convergent if the series $\sum|\alpha_i\beta_i|$, $i=1, 2, 3, \cdots, \infty$, is convergent, by a well known theorem about infinite products. But we have shown that $\Delta_m > |D_m|$. Hence D_m is absolutely convergent if the series $\sum|\alpha_i\beta_i|$, $i=1, 2, 3, \cdots, \infty$, is convergent and that is the *sufficient* condition for convergence of D_m.

Evidently it is also a *necessary* condition for convergence of the given determinant, for, if $\sum|\alpha_i\beta_i|$, $i=1, 2, 3, \cdots, \infty$, be divergent, D_m will not converge absolutely, since this series, with negative signs, is found in the expansion of D_m as may easily be verified.

Thus according to von Koch the absolute convergence of D_m depends only on second degree terms, $\alpha_i\beta_i$, in the development of the given determinant.

As a corollary to the fundamental theorem for normal determinants[2] it may be noticed here that D_m *is convergent if the series*

$$\alpha_1+\beta_1+\alpha_2+\beta_2+\cdots+\alpha_i+\beta_i+\cdots$$

is absolutely convergent.

But this theorem[3] does not give us the *necessary* condition for convergence of D_m as does von Koch's first lemma, for the series of *linear* terms

$$|\alpha_1|+|\beta_1|+|\alpha_2|+|\beta_2|+\cdots+|\alpha_i|+|\beta_i|+\cdots$$

does not occur in the development of D_m and hence we cannot assert that the divergence of the series

$$|\alpha_1|+|\beta_1|+|\alpha_2|+|\beta_2|+\cdots+|\alpha_i|+|\beta_i|+\cdots$$

makes D_m divergent as in the proof of von Koch's lemma. This can be shown by a simple example:
If

$$\alpha_1=\beta_1=1,\ \alpha_2=\beta_2=\tfrac{1}{2},\ \alpha_3=\beta_3=\tfrac{1}{3},\ \text{etc.},$$

the above series becomes

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[1] See Whittaker and Watson, loc. cit., p. 36 (§2.81).

[2] G. Kowalewski, *Einführung in die Determinantentheorie*, first edition, p. 372: "Wenn die unendliche Reihe $c_{11}+c_{12}+c_{21}+c_{13}+c_{22}+c_{31}+\cdots$ absolut konvergent ist, so existiert

$$\begin{vmatrix} 1+c_{11} & c_{12} & c_{13} & \cdots \\ c_{21} & 1+c_{22} & c_{23} & \cdots \\ c_{31} & c_{32} & 1+c_{33} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = \lim D_n."$$

[3] See the Bulletin de la Société Mathématique de France, vol. 14 (1886), p. 77.

$$\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\cdots$$

which is divergent; while Δ_m becomes

$$\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\cdots$$

which is convergent; hence D_m is convergent.

2. *Extensions.* Solution of homogeneous linear difference equations by means of infinite determinants[1] gives rise to the following determinant which is a generalization of the determinant of von Koch's first lemma:

$$\begin{vmatrix} 1 & b_0 & c_0 & 0 & 0 & 0 & \cdot & \cdot \\ a_1 & 1 & b_1 & c_1 & 0 & 0 & \cdot & \cdot \\ 0 & a_2 & 1 & b_2 & c_2 & 0 & \cdot & \cdot \\ 0 & 0 & a_3 & 1 & b_3 & c_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

This is absolutely convergent if the series $b_0a_1+b_1a_2+\cdots+b_{n-1}a_n+\cdots+c_0a_1a_2+c_1a_2a_3+\cdots+c_{n-1}a_na_{n+1}+\cdots$ *converges absolutely.*

Proof: Consider this determinant of order m:

$$D_m=\begin{vmatrix} 1 & b_0 & c_0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ a_1 & 1 & b_1 & c_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & a_2 & 1 & b_2 & c_2 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & a_3 & 1 & b_3 & c_3 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{m-1} & 1 \end{vmatrix};$$

and, to compare with it, construct the auxiliary determinant, Δ_m, in which all the elements are zeros except those of the principal diagonal; namely

$$\Delta_m=\begin{vmatrix} 1+b_0a_1 & 0 & 0 & 0 & \cdot & \cdot \\ 0 & 1+c_0a_1a_2 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 1+b_1a_2 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 1+c_1a_2a_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

[1] See the author's thesis, previously mentioned.

where b_ia_{i+1} and $c_ia_{i+1}a_{i+2}$ are all positive ($i=0, 1, 2, 3, \cdots$) and are *alternate factors* in the product of the diagonal elements.

Now in the expansion of Δ_m all the terms are positive and are formed by certain combinations of 2, 3, 4, $\cdots$, n letters at a time, except the first term which is unity, and the suffixes of b_ia_{i+1} and $c_ia_{i+1}a_{i+2}$ may be *repeated* in *any* combination; thus we shall have $b_0a_1+b_1a_2+\cdots+b_{m-2}a_{m-1}+c_0a_1a_2+c_1a_2a_3+\cdots+b_0a_1c_0a_1a_2+$ etc. where suffix 0 is *repeated* in the last term. There is no such repetition of subscripts in any term of the expansion of D_m. Hence Δ_m contains absolute values of all the terms of D_m plus additional terms all positive, since the upper limit of the suffixes in Δ_m is m while that of D_m is $m-1$ and some of the terms of D_m are negative. Hence Δ_m is greater than $|D_m|$.

Similarly the difference $(\Delta_{m+p}-\Delta_m)$ contains the absolute values of all the terms of the difference $(D_{m+p}-D_m)$ and other terms all positive. Consequently

$$|D_{m+p}-D_m| < \Delta_{m+p}-\Delta_m.$$

Hence the sequence D_m converges if the sequence Δ_m converges.

Now Δ_m is convergent if the infinite product of the main diagonal

$$\prod_{i=0}^{\infty}(1+b_ia_{i+1})(1+c_ia_{i+1}a_{i+2})$$

is convergent, since Δ_m and that infinite product are identical. But the product of the main diagonal is convergent if the series

$$\left\{\sum_{i=0}^{\infty}|b_ia_{i+1}|+\sum_{i=0}^{\infty}|c_ia_{i+1}a_{i+2}|\right\}$$

is convergent, by a well known theorem of infinite products. But we have shown that $\Delta_m > |D_m|$. Hence D_m is absolutely convergent if the series

$$\left\{\sum_{i=0}^{\infty}|b_ia_{i+1}|+\sum_{i=0}^{\infty}|c_ia_{i+1}a_{i+2}|\right\}$$

is convergent, and that is the *sufficient* condition for convergence of D_m. Obviously it is also the *necessary* condition for convergence of the given determinant; for, if

$$\left\{\sum_{i=0}^{\infty}|b_ia_{i+1}|+\sum_{i=0}^{\infty}|c_ia_{i+1}a_{i+2}|\right\}$$

be divergent, D_m will not converge absolutely, since this series, with signs changed, is found in the expansion of D_m as may readily be verified.

Thus the convergence of the determinant in question depends on the *quadratic* and *cubic* terms only in the development of the given determinant.

As a corollary to the fundamental theorem for normal determinants it may be noticed again that D_m *is convergent if the series*

$$b_0 + a_1 + c_0 + b_1 + a_2 + c_1 + b_2 + a_3 + c_2 + b_3 + a_4 + c_3 + b_4 + a_5 + c_4 + \cdots$$

is absolutely convergent. Here again, as on page 190, the converse is not true, i.e. the absolute convergence of Δ_m (and hence of D_m) does not imply the convergence of the above series of linear terms, as we may easily show by an example: Put

$$b_0 = c_0 = a_1 = 1, \quad b_1 = c_1 = a_2 = \tfrac{1}{2}, \quad b_2 = c_2 = a_3 = \tfrac{1}{3}, \quad b_3 = c_3 = a_4 = \tfrac{1}{4},$$

generally,

$$b_i = c_i = a_{i+1} = \frac{1}{i}. \quad (i = 0, 1, 2, \cdots).$$

Then the above series becomes

$$1 + 1 + 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} + \cdots,$$

which is divergent; while Δ_m becomes

$$\left\{\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\cdots + \left(1+\frac{1}{1^2.2}\right)\left(1+\frac{1}{2^2.3}\right)\left(1+\frac{1}{3^2.4}\right)\cdots\right\}$$

which is obviously convergent.

Thus the convergence of the above series of linear terms implies the absolute convergence of D_m, while the convergence of D_m does not imply anything about the series.

Generally, the solution of homogeneous linear difference equations of order n by the said method gives rise to the following determinant which is the most general extension to the determinant of von Koch's first lemma:

$$D_m = \begin{vmatrix}
1 & a_0^{(2)} & a_0^{(3)} & a_0^{(4)} & \cdots & a_0^{(m)} & 0 & \cdots & 0 \\
a_1^{(1)} & 1 & a_1^{(2)} & a_1^{(3)} & a_1^{(4)} & \cdots & a_1^{(m)} & 0 & \cdots & 0 \\
0 & a_2^{(1)} & 1 & a_2^{(2)} & a_2^{(3)} & a_2^{(4)} & \cdots & a_2^{(m)} & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & a_{m-1}^{(1)} & 1 & \cdots & \cdots & \cdots & a_{m-1}^{(m)} & 0
\end{vmatrix}$$

As before, we can construct an auxiliary determinant Δ_m in which the elements are zeros except those of the leading diagonal and show that (since Δ_m contains all the moduli of the terms of D_m plus additional terms all positive) $\Delta_m > |D_m|$ and $\Delta_{m+p} - \Delta > |D_{m+p} - D_m|$. Hence if Δ_m be convergent D_m will be also; and the convergence of the latter will depend on the quadratic, cubic,

$\cdots$, $(n-1)$-ic terms in the development of the given determinant; while if we apply Poincaré's fundamental theorem for normal determinants we can show that a *sufficient* condition of convergence of the above determinant is that the series

$$a_0^{(2)} + a_1^{(1)} + a_0^{(3)} + a_0^{(4)} + a_1^{(2)} + a_2^{(1)} + \text{etc.}$$

be absolutely convergent. But the converse is not true as before, and this can be shown by constructing special examples as was done in the first and second parts of this paper.

It is to be noted that not only such determinants as those considered in this paper are convergent, but the same is true of their minors and of all determinants obtained from them by replacing a finite number of their rows or columns by a set of quantities which are numerically less than some constant.

A FEW THEOREMS RELATING TO THE RHIND MATHEMATICAL PAPYRUS

By G. A. MILLER, University of Illinois

One of the most striking features of the *Rhind Mathematical Papyrus* is the table at the beginning thereof in which 2 divided by the various odd numbers from 5 to 101 is expressed in each case as the sum of the reciprocals of *different* natural numbers. Hence this table illustrates a very special case of the following theorem: *An arbitrary rational number can be expressed as the sum of the reciprocals of a finite set of distinct integers having the same sign and such that the smallest absolute value of the integers of the set is arbitrarily large and that the number of these integers exceeds any assigned natural number.* Hence it may be said that the integers of such a set can be selected in a multiply infinite number of ways.

To prove this theorem we may consider the infinite series formed by the reciprocals of the natural numbers in order. This series was proved to be divergent by Mengoli in 1650 and is commonly known as the harmonic series. It is obvious that we may confine our attention to the case when the arbitrary rational number n under consideration is positive since the proof of the case when it is negative can be derived from the proof when it is positive by merely changing the signs of all the numbers involved therein. When n represents an arbitrary positive rational number we may begin with any term of the harmonic series which satisfies the single condition that it does not exceed n, and add to it a sufficient number of terms which follow it to obtain a sum s which does not exceed n but that $n-s$ is less than any of the terms of the harmonic series thus added. It is well known that the latter can always be done in an infinite number of ways when $s \neq n$.

When $s=n$ the theorem under consideration has thus been proved for this particular value of n. When s is less than n we may find the common fraction

which is equal to $n-s$ and reduce it to lowest terms a/b. Since a and b are relatively prime it is always possible to find two natural numbers, x and y, $y<b$ so that $ay-bx=1$, and hence

$$\frac{a}{b}-\frac{x}{y}=\frac{1}{by}.$$

If $x=1$ our problem is solved since the unit fractions $1/y$ and $1/(by)$ are then distinct from each other and also from each one of the unit fractions which were added to obtain s. If $x>1$ we can find in a similar way another fraction x_1/y_1 such that $y_1<y$ and that

$$\frac{x}{y}-\frac{x_1}{y_1}=\frac{1}{yy_1},$$

where $yy_1<by$. When $x_1=1$ then $1/y_1$ is distinct from all of the unit fractions which have thus far been added and it is larger than $1/(yy_1)$. Since the absolute values of the denominators $y, y_1, \cdots$ are decreasing integers and all the fractions involved in these equations are proper fractions it is obvious that this process can be continued until $x_k=1$, and hence the theorem under consideration has been proved.

The history of elementary mathematics can frequently be enriched by entering more deeply into the mathematical questions involved than could have been done at the period to which this history relates since mathematical developments are frequently inspired by ideas whose bearings are only partially understood at the time. In particular, while the ancient Egyptians could not have proved the theorem noted above a knowledge of its existence shows us now that their efforts to express numbers as the sum of different unit fractions were likely to lead to an unexpected richness of results. It should also be noted that only a finite number of different unit fractions is needed to represent as such a sum any rational number while this is not always the case when such a number is expressed as the sum of different powers of a single number, such as powers of 10.

This fact may help to explain the late adoption of the positional decimal system, especially as regards fractions. The theorem noted above becomes especially significant when it is observed that no pre-Grecian civilization is known to have extended the number concept beyond that of the positive rational numbers. Hence this theorem seems to apply to all of their numbers and it may help to explain the popularity of unit fractions among them even if only special cases of this theorem could be understood at that time. The fact that all the unit fractions which appear in the table noted above are distinct indicates a theoretic interest on the part of the ancient Egyptians which deserves emphasis here since mathematical historians have frequently directed attention to the practical trend of the ancient Egyptian mathematics. It should also be noted here that J. J. Sylvester called attention to a different line of mathematical developments which he associated with the table under consideration.[1]

[1] American Journal of Mathematics, volume 3 (1880), pages 332 and 388.

The given theorem establishes the fact that a necessary and sufficient condition that a number is rational is that it is the sum of a finite number of the distinct terms of the harmonic series, or of the series obtained by changing all the signs thereof. Every rational integer is also the sum of a finite number of distinct powers of 2, or of such powers of 2 with all of their signs changed. This may explain why multiplication was so commonly performed by successive doubling in the *Rhind Mathematical Papyrus*. It is not likely that the ancient Egyptians actually proved this theorem but their confidence therein was probably established by numerous examples. Just as multiplication was commonly performed in this work by successive doubling so division was usually accomplished by successive halving. Their most important measures were divided into halves, quarters, eighths, etc., and it has been suggested[1] that there may have been a stage when instead of the general unit fractions the ancient Egyptians used only powers of $\frac{1}{2}$.

From what was stated above it results that it is not possible to express every positive fraction in terms of a finite number of different powers of $\frac{1}{2}$. Even the fraction $\frac{2}{3}$ which was extensively used by the ancient Egyptians and others can not be thus expressed. The mathematics of conscious approximation is naturally a later development than the mathematics of accuracy, or supposed accuracy, since the former requires a consideration of the desirable degrees of approximation for various needs. It does not seem likely therefore that when the ancient Egyptians assumed the area of a circle to be equal to that of a square whose side is equal to 8/9 of the diameter of the circle they were conscious of the fact that this result is only approximately true. At any rate, it is very interesting to note that by the use of the sum of a finite number of unit fractions they could express any of their given fraction exactly, and that the table noted above contains some very small fractions of this type, for instance 1/890, which could probably have been omitted if their interest had been confined to practical applications. It is much smaller than the relative error resulting from their rule for finding the area of the circle.

Another theorem which is suggested by work in the *Rhind Mathematical Papyrus* but was probably not fully understood at the time is that the square represents an upper limit of the area of a rhombus with a given side. In the selection of units of area and units of volume squares and cubes were commonly chosen and thus certain numerical relations were established between geometrical quantities having different dimensions. We may now see in this decided steps towards the later subject of analytic geometry and towards a closer union between numbers and geometric figures, which are clearly the most fundamental concepts of all mathematics. The difficulty of determining whether the area of an isosceles triangle with a relatively short base was actually supposed to be the product of half the base and the altitude or half the base and a side may have been partly due to a lack of noting the difference between the area

[1] Peet, *Rhind Mathematical Papyrus* (1923), page 15.

of a square and that of a rhombus having an equal side and an angle almost equal to a right angle.

While the operation of addition is simpler than that of multiplication when dealing with integers the reverse is true when dealing with common fractions. In view of the very early use of such fractions, as is evidenced by special symbols for some of the most common ones, such as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{4}$, it is easy to see that the addition of fractions may have greatly influenced the selection of certain numbers which were especially favored. For instance, in adding $\frac{1}{2}$ and $\frac{1}{3}$ it is desirable to reduce them to the common denominator 6, and this may explain, at least in part, why the number 6 played such a prominent rôle in ancient mathematics. Similarly, in adding $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ it is convenient to reduce them to the common denominator 12, and in adding $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ the common denominator 60 is most convenient. This common denominator will suffice to add also $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{1}{6}$, and hence it is especially significant, being the first which does not require any change when our list of reciprocals is extended by unity.

It is always interesting to find unifying principles in the history of elementary mathematics. The addition of certain unit fractions as found in the *Rhind Mathematical Papyrus* seems to furnish such a principle. Doubtless other principles were potent in securing for the numbers 6, 12, and 60 the favor which they enjoyed in ancient times, as is evidenced by the sexagesimal system of numeration and the emphasis on fractions whose denominators are powers of 12 among the Romans. The fact that we have not yet found among the pre-Grecian mathematicians, nor even among the ancient Greeks, a special word for common denominator, or lowest common denominator, would seem to indicate that their mathematical ideas along this line were not as definite as one might have expected from their advancement along other lines. It must, however, be remembered that even indefinite ideas are sometimes potent, and these constitute often the most difficult problems of mathematical history.

While the rule which the ancient Egyptians employed to find the area of a circle is not accurate it implies the fundamental theorem that the areas of two circles are to each other as the squares of their diameters. This theorem is said to have been proved by Hippocrates and was extended by the ancient Greeks to various similar plane figures as well as to similar solids. It should not be assumed that the ancient Egyptians formulated this theorem but it is interesting to note how closely they approached to it in the *Rhind Mathematical Papyrus*. By the consideration of such approaches the history of elementary mathematics constitutes a powerful incentive for the study of certain subjects of mathematics itself and it enables us also to see more clearly the limitations of the attainments of ancient peoples. A study of these limitations is as important in securing clear historical insight as the study of the advances made in early times, since mountain tops alone fail to give a complete picture of a country.

A CERTAIN POLYNOMIAL EXPANSION

By FLORA STREETMAN and L. R. FORD, Rice Institute

1. *Introduction.* The present paper contains an investigation of the expansion of a function $f(z)$, analytic at the origin, in a series

$$f(z) = P_0(z) + P_1(z) + P_2(z) + \cdots, \tag{1}$$

where $P_n(z)$ is a polynomial of degree n at most whose explicit form is

$$P_n(z) = \frac{1}{(1+h)^{n+1}}\left[h^n f(0) + nh^{n-1}f'(0)z + \frac{n(n-1)}{(2!)^2}h^{n-2}f''(0)z^2 + \frac{n(n-1)(n-2)}{(3!)^2}h^{n-3}f'''(0)z^3 + \cdots + \frac{1}{n!}f^{(n)}(0)z^n\right], \tag{2}$$

h being a positive constant. If $f(z)$ has the series expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots,$$

(2) may be written in the alternative form

$$P_n(z) = \frac{h^n}{(1+h)^{n+1}}\left[a_0 + na_1\frac{z}{h} + \frac{n(n-1)}{2!}a_2\left(\frac{z}{h}\right)^2 + \cdots + a_n\left(\frac{z}{h}\right)^n\right].$$

A region S about the origin in which the series converges and represents the function is constructed. This region is larger than the circle of convergence of Maclaurin's series for the function unless the circle of convergence is the natural boundary; hence the series (1) provides a formula for the analytic continuation of the function beyond the limits of the circle of convergence.

2. *Derivation of the Series.* Let Γ be a regular curve enclosing the origin, within which $f(z)$ is analytic and on which $f(z)$ is continuous. We have from Cauchy's formula for z in Γ

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)dt}{t-z}. \tag{3}$$

Writing $t-z=(1+h)t-(z+ht)$, we find

$$\frac{1}{t-z} = \frac{1}{(1+h)t} + \frac{z+ht}{(1+h)^2t^2} + \cdots + \frac{(z+ht)^m}{(1+h)^{m+1}t^{m+1}} + \frac{(z+ht)^{m+1}}{(1+h)^{m+1}t^{m+1}(t-z)};$$

whence, substituting in (3),

$$f(z) = P_0(z) + P_1(z) + \cdots + P_m(z) + R_m(z),$$

where

$$P_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(z+ht)^n}{(1+h)^{n+1}t^{n+1}}f(t)dt,$$

$$R_m(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(z+ht)^{m+1}}{(1+h)^{m+1}t^{m+1}(t-z)}f(t)dt.$$

It is clear that $P_n(z)$ is a polynomial of degree n at most. Its value is

$$P_n(z) = \frac{1}{n!} \cdot \frac{1}{(1+h)^{n+1}} \frac{d^n}{dt^n}[(z+ht)^n f(t)]_{t=0}.$$

On differentiating n times and setting $t=0$, we have the polynomial appearing in (2).

3. *The Major Circle.* The major circle for any point z is the circle with center $-hz$ and radius $(1+h)|z|$. (See M, Figure 1, constructed for $h=1$.)

The major circles for all points t on Γ are now constructed. Let σ be a region whose points lie within all such major circles. A sufficiently small neighborhood of the origin, for example, has this property.

Theorem I. *In any region σ' wholly within σ the series* (1) *converges uniformly to the function $f(z)$.*

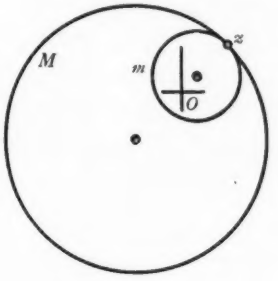

FIG. 1. Major and Minor Circles.

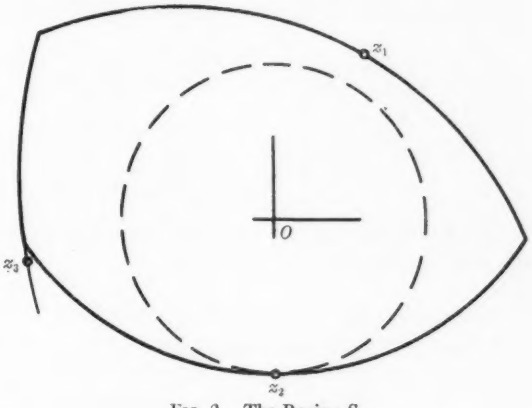

FIG. 2. The Region S.

If z lies in M_t, the major circle for a point t on Γ, its distance from the center $-ht$ of M_t is less than the distance from t to $-ht$; and

$$|(z+ht)/(t+ht)| < 1.$$

It can be shown that for any z in σ' and any t on Γ

$$|(z+ht)/(t+ht)| < r < 1.$$

Then since

$$\left|\frac{1}{2\pi i}\frac{f(t)}{t-z}\right| < K, \text{ a constant,}$$

we have

$$|R_m(z)| < r^{m+1} K\gamma/2\pi,$$

where γ is the length of Γ, and the remainder $R_m(z)$ approaches zero uniformly as m approaches infinity for all values of z in σ'.

4. *The Region S.* Let a ray, that is, a half-line, issue from the origin. We proceed from the origin along this ray until a singularity of the function is encountered, if any. We then construct the major circle for this singularity. We do this for each ray. The region within all the major circles so constructed will be called the region S.

In Figure 2 we show S for a function with three singularities, taking $h=1$. The broken circle is the circle of convergence of Maclaurin's series. It is clear that all points of the circle of convergence at which the function is analytic are interior points of S.

5. *The Minor Circle.* The minor circle for any point z is the circle with center $h(1+2h)^{-1}z$ and radius $(1+h)(1+2h)^{-1}|z|$. (See m, Figure 1.)

Theorem II. *If z_1 lies inside, on, or outside the minor circle of z_2, then z_2 lies respectively outside, on, or inside the major circle of z_1.*

If z_1 lies inside, on, or outside the minor circle m_2 of z_2, its distance from the center of m_2 is respectively less than, equal to, or greater than the radius of m_2. We have then

$$\left| z_1 - \frac{h}{1+2h} z_2 \right| \gtreqless \frac{1+h}{1+2h} |z_2| ,$$

using the upper, middle, or lower sign when z_1 lies respectively inside, on, or outside m_2. Letting

$$z_1 = x + iy, \quad z_2 = m + in,$$

and substituting in the above, we have

$$\left| x + iy - \frac{h}{1+2h}(m + in) \right| \gtreqless \frac{1+h}{1+2h} |m + in| ,$$

which gives

$$[(1+2h)x - hm]^2 + [(1+2h)y - hn]^2 \gtreqless (1+h)^2(m^2+n^2).$$

On expanding and rearranging terms, we have

$$(m+hx)^2 + (n+hy)^2 \gtreqless (1+h)^2(x^2+y^2).$$

Thus z_2 lies outside, on, or inside the circle with center

$$m = -hx, \quad n = -hy, \quad m + in = -hz,$$

and radius

$$(1+h)(x^2+y^2)^{1/2} = (1+h)|z_1| ;$$

that is, the major circle of z_1.

The region σ of Section 3 can now be given a new interpretation. It is a region such that the minor circle of any of its interior points lies within Γ.

Theorem III. *A necessary and sufficient condition that $f(z)$ be analytic within and on the minor circle of z_2 is that z_2 lie in S.*

(1). If z_2 lies in S, then $f(z)$ is analytic in and on the minor circle of z_2.

From Theorem II we know that if z_2 lies inside the major circle of z_1, z_1 lies outside the minor circle of z_2. Thus since z_2 lies within S, it lies within the major circles of all the singularities, and hence all the singularities lie outside of the minor circle of z_2.

(2). If $f(z)$ is analytic in and on m_2, the minor circle of z_2, then z_2 lies in S.

Since $f(z)$ is analytic in and on m_2, all singular points lie outside m_2. Hence by Theorem II, z_2 lies inside the major circles of all singularities, that is, in the region S.

It will be observed that the minor circle of a point S is not necessarily wholly within S.

6. *Convergence of the Series in S.* The proof of the convergence of the series rests on the following proposition.

Theorem IV. *For any point z_2 in S, there exists a circle, enclosing z_2 and the origin, within and on which $f(z)$ is analytic, and such that the major circles of all points on the circle enclose a region in S about the point z_2.*

A circle Γ concentric with the minor circle for z_2, m_2, and of slightly larger radius can be constructed such that $f(z)$ is analytic in and on Γ, since we know $f(z)$, by Theorem III, to be analytic in and on m_2.

Since the center and radius of the minor circle are continuous functions of z, it follows that for z in a sufficiently small circle σ about z_2, the minor circle of z lies in Γ. Any point t on Γ is outside the minor circle of any point of σ; hence, from Theorem II, any point of σ is within the major circle of t.

We know by Theorem I that the series of polynomials converges uniformly to $f(z)$ in σ. Since we can find such a circle σ for any point within S, we can state the following theorem:

Theorem V. *The series of polynomials*

$$P_0(z) + P_1(z) + \cdots + P_n(z) + \cdots$$

converges to $f(z)$ in the region S.

The following theorem is also readily established.

Theorem VI. *The convergence of the series of polynomials is uniform in any region S' lying wholly within S.*

About each interior or boundary point of S' there exists a circle in which the series converges uniformly. Since by the Heine-Borel theorem S' can be covered by a finite number of these circles, the convergence is seen to be uniform for the region S'.

INTEGRAL SOLUTIONS OF $ax^3+by^3=az^3+bt^3$

By PERRY A. CARIS, University of Pennsylvania

1. *Introductory Remarks.* The Diophantine equation

$$ax^3 + by^3 = az^3 + bt^3, \tag{1}$$

where a and b are any non-zero integers seems never to have been treated. The special case $a=b=1$ has been solved by Euler,[1] Binet,[2] Carmichael[3] and others. The case $a=2$, $b=1$, has been treated by Amsler[4] and Gérardin.[4]

It is the purpose of this paper to find non-trivial solutions of (1) no matter what permissible values a and b may have. Some trivial solutions are obvious. For example, $x=z$, $y=t$. But it is perhaps not so obvious that $xy=tz$ would imply a trivial solution. To make this clear let $t=xy/z$ in (1). Then (1) reduces to

$$ax^3z^3 + by^3z^3 = az^6 + bx^3y^3.$$

That is

$$az^3(x^3 - z^3) = by^3(x^3 - z^3)$$

which, since $x \neq z$, implies $az^3=by^3$ and therefore also $ax^3=bt^3$ which is trivial. Hence these assumptions are made: $x \neq z$ and $xy \neq tz$.

2. *Solution of the given equation.* Write (1) in the form

$$a(x^3 - z^3) = b(t^3 - y^3).$$

This equation will be satisfied if

$$ap(x - z) = bq(t - y),$$

$$q(x + \tfrac{1}{2}z - \tfrac{1}{2}z\sqrt{(-3)}) = (r + s\sqrt{(-3)})(t + \tfrac{1}{2}y - \tfrac{1}{2}y\sqrt{(-3)}),$$

$$(r + s\sqrt{(-3)})(x + \tfrac{1}{2}z + \tfrac{1}{2}z\sqrt{(-3)}) = p(t + \tfrac{1}{2}y + \tfrac{1}{2}y\sqrt{(-3)}).$$

These three equations imply the following in real numbers:

$$\begin{aligned} apx + bqy - apz - bqt &= 0, \\ 2qx - (r + 3s)y + qz - 2rt &= 0, \\ (s - r)y + qz + 2st &= 0, \\ 2rx - py + (r - 3s)z - 2pt &= 0, \\ 2sx - py + (r + s)z &= 0. \end{aligned} \tag{2}$$

From the first three of these equations $x:y:z:t=$

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[1] Dickson's *History of the Theory of Numbers*, vol. 2, pp. 552–554.

[2] *Ibid.*, pp. 554–555.

[3] Carmichael's *Diophantine Analysis*, pp. 62–66.

[4] Dickson's *History of the Theory of Numbers*, vol. 2, p. 562.

$$\begin{vmatrix} bq & -ap & -bq \\ -r-3s & q & -2r \\ s-r & q & 2s \end{vmatrix} : \begin{vmatrix} ap & -ap & bq \\ 2q & q & 2r \\ 0 & q & -2s \end{vmatrix} : \begin{vmatrix} ap & bq & -bq \\ 2q & -r-3s & -2r \\ 0 & s-r & 2s \end{vmatrix}$$

$$: \begin{vmatrix} ap & -bq & -ap \\ 2q & r+3s & q \\ 0 & r-s & q \end{vmatrix},$$

whence

$$\begin{aligned} x &= \lambda_1(bq^2r + 3bq^2s - apr^2 - 3aps^2), \\ y &= \lambda_1(bq^3 - 3apqs - apqr), \\ z &= \lambda_1(bq^2r - 3bq^2s - apr^2 - 3aps^2), \qquad \lambda_1 \neq 0. \\ t &= \lambda_1(bq^3 + 3apqs - apqr). \end{aligned} \tag{3}$$

The substitution of these values in the fourth and the fifth equations of (2) gives

$$(bq^2 - apr + aps)(r^2 + 3s^2 - pq) = 0$$

and

$$(bq^2 - apr - 3aps)(r^2 + 3s^2 - pq) = 0,$$

respectively. These will both be satisfied if

$$pq = r^2 + 3s^2. \tag{4}$$

Now the x of (3) equals

$$\lambda_1[bq^2r + 3bq^2s - ap(r^2 + 3s^2)] = \lambda_1(bq^2r + 3bq^2s - ap^2q). \tag{5}$$

Similarly, the z of (3) equals

$$\lambda_1(bq^2r - 3bq^2s - ap^2q). \tag{6}$$

If $q=0$, $x=z=0$ and the result is trivial. Hence $q\neq 0$. Let $\lambda_1=\lambda/q$ and combine (5) and (6) with y and t of (3) and obtain the solution

$$\begin{aligned} x &= \lambda(bqr + 3bqs - ap^2), \\ y &= \lambda(bq^2 - 3aps - apr), \\ z &= \lambda(bqr - 3bqs - ap^2), \\ t &= \lambda(bq^2 + 3aps - apr), \end{aligned} \tag{7}$$

with the condition $pq=r^2+3s^2$.

Consider now the converse problem. Let x, y, z, t be non-trivial values satisfying (1). It remains to show that λ, p, q, r, s can be found to give precisely these values of x, y, z, t. From the second and the third equations of (2),

$$q:r:s = [(2y + t)^2 + 3t^2]:(2xy + 4xt + 4yz + 2tz):(2xy - 2tz).$$

If q, r, s be taken equal to these values, then (4) will be satisfied if

$$p = (r^2 + 3s^2)/q = (2x + z)^2 + 3z^2.$$

Now from (7), $x-z=6bq\lambda s$, whence

$$\lambda = \frac{x - z}{48b(y^2 + yt + t^2)(xy - tz)}.$$

Since the supposed solution is non-trivial, λ is neither 0 nor ∞. Moreover these values of λ, p, q, r, s will satisfy (7). Consider, for example, $\lambda(bqr+3bqs-ap^2)$. This equals

$$(8)\quad \frac{x - z}{48b(y^2 + yt + t^2)(xy - tz)}[8b(y^2 + yt + t^2)(xy + 2xt + 2yz + tz) + 24b(y^2 + yt + t^2)(xy - tz) - 16a(x^2 + xz + z^2)^2]$$

$$= \frac{x - z}{3(xy - tz)}\left[2xy + xt + yz - tz - \frac{a(x^2 + xz + z^2)}{b}\left(\frac{x^2 + xz + z^2}{y^2 + yt + t^2}\right)\right].$$

But, by hypothesis,

$$a(x^3 - z^3) = b(t^3 - y^3).$$

That is

$$a(x - z)(x^2 + xz + z^2) = b(t - y)(y^2 + yt + t^2),$$

whence

$$\frac{x^2 + xz + z^2}{y^2 + yt + t^2} = \frac{b(t - y)}{a(x - z)}.$$

Then (8) can be continued thus:

$$\frac{x - z}{3(xy - tz)}\left[2xy + xt + yz - tz - \frac{(t - y)(x^2 + xz + z^2)}{x - z}\right]$$

$$= \frac{1}{3(xy - tz)}(3x^2y - 3xtz) = x.$$

The work for y, z, t is similar.

The solution (7) may be expressed in terms of independent parameters by giving explicit values to p, q, r, s. Let $p=c^2+3d^2$; $q=f^2+3g^2$. Then by the method of composition of binary quadratic forms,

$$r = cf \pm 3dg;\ s = cg \mp df.$$

A SIMPLIFIED INTEGRAL TEST FOR THE CONVERGENCE OF INFINITE SERIES

By RAYMOND W. BRINK, University of Minnesota

In other papers[1] the author has presented certain integral tests for the convergence and divergence of infinite series. Such tests are interesting not only because they can be used for testing types of series which are very difficult to examine by other methods, but also because, through the natural connection between integration and summation, they offer a simple and attractive means of unifying and establishing many tests of other kinds.

DuBois-Reymond[2] gave the name "tests of the first kind" to series tests which make direct use of the general term of the series itself. And "tests of the second kind" are those which use the ratio of the general term to the preceding term. In this category are the d'Alembert test, which every student of Calculus knows as "the" ratio test, the first test of Raabe, and the sequence of tests of de Morgan and Bertrand.

In a similar way, the familiar Maclaurin-Cauchy integral test,[3] which requires the use of a function $u(x)$ where $u(n)$ is the general term of the series, may be called an "integral test of the first kind." And an "integral test of the second kind" is one in which an analogous rôle is played by a function $r(x)$, where $r(n)$ is the ratio of the nth term to the preceding term.

Hitherto the most generally useful integral test of the second kind was given by the author in one of the papers mentioned.[4] Its statement and proof are simple for convergence, but rather awkward for the divergence test. The purpose of the present paper is to give a modified form of the test—a form that is as widely applicable as the earlier one and that is simpler to state and to establish. Two proofs will be given, first a very short proof based on a theorem of the earlier paper and second a direct proof so simple as to be available to good students of intermediate grade. The condition of limited variation which is imposed on the ratio-function $r(x)$ is sufficiently general for all cases met in practice, and, of course, includes such cases as that in which $r(x)$ increases monotonically toward the limit 1.

Theorem: *Given a series*

$$u_0 + u_1 + u_2 + \cdots, \quad (u_n > 0, n \geqq \mu).$$

Let $r(x)$ be a function with a derivative $r'(x)$ satisfying the following conditions for values of x greater than or equal to a certain integer μ:

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[1] Transactions of the American Mathematical Society, vol. 19 (1918), p. 186. Annals of Mathematics, Ser. 2, vol. 21 (1919), p. 39.

[2] Journal für Mathematik, vol. 76 (1891), p. 61.

[3] Maclaurin, *Treatise on Fluxions* (1742), vol. 1, p. 289. Cauchy, *Exercises de Mathématique* (1827), vol. 2, p. 221.

[4] Transactions of the American Mathematical Society, vol. 19 (1918), p. 195.

$$r(n) = r_n = u_{n+1}/u_n, \tag{1}$$

$$0 < A \leqq r(x) \leqq B, \tag{2}$$

$$\int_\mu^x | r'(x) | \, dx < D, \; x > \mu. \tag{3}$$

Then the convergence of the integral

$$\int_\mu^\infty e^{R(\mu,x)}dx, \text{ where } R(\mu, x) = \int_\mu^x [r(x) - 1]dx, \tag{A}$$

is sufficient for the convergence of the given series; and the divergence of the integral

$$\int_\mu^\infty e^{\rho(\mu,x)}dx, \text{ where } \rho(\mu, x) = \int_\mu^x [1 - 1/r(x)]dx, \tag{B}$$

implies the divergence of the series.

Under the conditions (1), (2) and (3) the convergence of the integral

$$\int_\mu^\infty e^{\sigma(\mu,x)}dx, \text{ where } \sigma(\mu, x) = \int_\mu^x \log r(x)dx, \tag{C}$$

is known[1] to be necessary and sufficient for the convergence of the series. Then the theorem follows from the familiar inequalities[2]

$$\left.\begin{aligned} \log r(x) &\leqq r(x) - 1 \\ \log r(x) &\geqq 1 - 1/r(x) \end{aligned}\right\}, \; 0 < r(x).$$

For the convergence of (A) is sufficient for the convergence of (C) and therefore for that of the series. And if (B) diverges, so do (C) and the series.

We will now establish the theorem without the use of the integral (C). Set

$$d_n = \int_n^{n+1} [r(x) - 1]dx - (r_{n+1} - 1) = \int_n^{n+1} [r(x) - r_{n+1}]dx. \tag{4}$$

Integrating by parts, we get

$$d_n = -\int_n^{n+1} (x - n)r'(x)dx \tag{5}$$

so that

$$| d_n | \leqq \int_n^{n+1} | r'(x) | \, dx. \tag{6}$$

Therefore, by (3),

$$\sum_{n=\mu} | d_i | < D, \; n > \mu, \text{ and } \sum_{i=\mu}^{n} d_i > -D.$$

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[1] Transactions of the American Mathematical Society, vol. 19 (1918), p. 188, Theorem II.

[2] See, for instance, Pringsheim, Mathematische Annalen, vol. 35, p. 317.

If we use a bar to indicate that the integrals under it have the same integrand, for any value of x for which $n \leqq x \leqq n+1$,

$$R(\mu, x) = \int_{\mu}^{x} [r(x) - 1]dx = \overline{\int_{\mu}^{\mu+1} + \int_{\mu+1}^{\mu+2} + \cdots + \int_{n-1}^{n} + \int_{n}^{x}} [r(x) - 1]dx$$

$$= \sum_{i=\mu}^{n-1} d_i + \sum_{i=\mu}^{n-1} (r_{i+1} - 1) + \int_{n}^{x} [r(x) - 1]dx \tag{7}$$

$$> -D - 1 + (r_{\mu+1} - 1) + (r_{\mu+2} - 1) + \cdots + (r_n - 1).$$

Therefore

$$\int_{n}^{n+1} e^{R(\mu,x)}dx > e^{-D-1} \cdot e^{(r_{\mu+1}-1)} \cdot e^{(r_{\mu+2}-1)} \cdots e^{(r_n-1)}. \tag{8}$$

Since $e^{z-1} \geqq z$,

$$\int_{n}^{n+1} e^{R(\mu,x)}dx > e^{-D-1} \cdot r_{\mu+1} \cdot r_{\mu+2} \cdots r_n = \frac{e^{-D-1}}{u_{n+1}} u_{\mu+1}, \ (\mu \leqq n). \tag{9}$$

Therefore if the integral (A) converges, the series

$$\sum_{n=\mu}^{n} u_n$$

also converges.

To establish the test for divergence we proceed in a similar way, this time setting

$$d_n = \int_{n}^{n+1} [1 - 1/r(x)]dx - (1 - 1/r_{n+1}),$$

and using the inequality

$$e^{1-(1/z)} \leqq z, \ (0 < z).$$

In its modified form, this integral test can be applied to many special series of the normal type

$$c + cr_0 + cr_0r_1 + cr_0r_1r_2 + \cdots$$

where $r_n = r(n)$, $r(x)$ being a known simple function. For example, consider the series

$$1 + (1 - k) + (1 - k)\left(1 - \frac{k}{2}\right) + (1 - k)\left(1 - \frac{k}{2}\right)\left(1 - \frac{k}{3}\right) + \cdots. \tag{10}$$

Since the function $r(x)=1-(k/x)$ is finite and monotonic, the conditions for the test are satisfied. Then, if $k>1$, the series converges, for

$$R(\mu, x) = -\int_{\mu}^{x}(k/x)dx = k\log(c/x)$$

and the integral

$$\int_{\mu}^{\infty} e^{R(\mu,x)}dx = \int_{\mu}^{\infty}(c/x)^k dx$$

converges. Similarly, if $k\leqq 1$, the series diverges, for the integral

$$\int_{\mu}^{\infty} e^{\rho(\mu,x)}dx = \int_{\mu}^{\infty}\frac{c\,dx}{(x-k)^k}$$

diverges.

Moreover, as has already been suggested, it is possible to give a discussion, based on this integral test, of the tests of the logarithmic scale and of most of the other tests of the second kind. For example, to establish Raabe's first test,[1] which depends on the value of $l=\lim_{n=\infty}\omega_n$, where $r_n=1-(\omega_n/n)$, we need only to compare the given series with the series (10) of the preceding example, taking k between l and 1.

The extraordinary power of the present integral test and the simplicity of its conditions as compared with the divergence test of the earlier article is shown by the ease with which it can be used to establish the sequence of Bertrand's delicate tests:[2]

If

$$\frac{1}{r(x)} = 1+\frac{1}{x}+\frac{1}{x\cdot l_1x}+\cdots+\frac{1}{x\cdot l_1x\cdots l_{k-1}x}+\frac{\omega(x)}{x\cdot l_1x\cdots l_kx},$$

and if $\lim_{x=\infty}\omega(x)=l$, *the corresponding series converges when* $l>1$ *and diverges when* $l<1$. Thus suppose that $l<1$. Take λ between l and 1, and compare the given series with a series for which

$$\frac{1}{r(x)} = 1+\frac{1}{x}+\cdots+\frac{1}{x\cdot l_1x\cdots l_{k-1}x}+\frac{\lambda}{x\cdot l_1x\cdots l_kx}.$$

For this series,

$$\int_{\mu}^{\infty} e^{\rho(x)}dx = c\int_{\mu}^{\infty}\frac{dx}{x\cdot l_1x\cdot l_2x\cdots l_{k-1}x(l_kx)^{\lambda}},$$

a divergent integral. The given series therefore diverges.

The present integral test can be extended to multiple series and also to series of functions. In the latter case, uniform convergence of the integral implies uniform convergence of the corresponding series.

[1] Journal für Mathematik, vol. 11, p. 309.

[2] Journal de Mathématiques, vol. 7, p. 42.

ON A CURVE ASSOCIATED WITH A TRIANGLE

By JAMES H. WEAVER, Ohio State University

Let there be a triangle $A_1A_2A_3$ and a point P in the plane of the triangle (see Fig. 1) determined as follows:

$$\angle A_iPA_j = \angle (m\pi + kA_n) \quad (i, j, n, = 1, 2, 3,\ i \neq j \neq n). \tag{1}$$

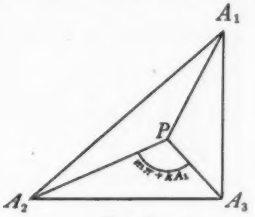

FIG. 1

From (1) we have, by addition, $3m+k=2$. If we transform (1) by means of

$$3m + k = 2,\ A_n = \tfrac{1}{3}\pi - A_n', \tag{2}$$

we obtain

$$\angle A_iPA_j = \angle (\tfrac{2}{3}\pi - kA_n'). \tag{3}$$

If now k is allowed to vary the point P will describe a curve C.

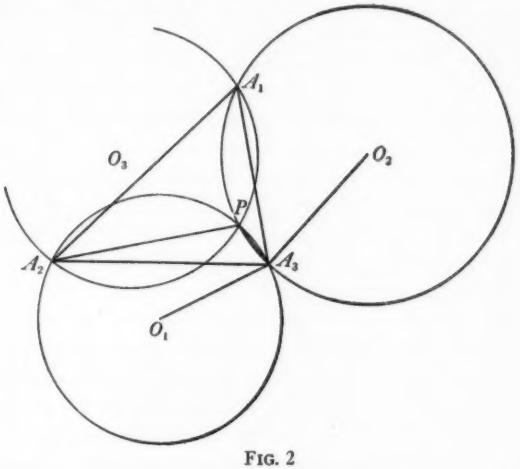

FIG. 2

Points on the curve C may be constructed by drawing on A_iA_j segments of circles which will contain the angles $\frac{2}{3}\pi - kA_n'$. (See Fig. 2). Equations (3) show that if $A_n' = 0$, the circle through A_iPA_j will be fixed. We therefore have this theorem:

If one of the angles of the triangle is $\frac{1}{3}\pi$ the locus of P is a circle passing through the other two vertices.

It is obvious that the curve will pass through a vertex only when one of the circles is a circumcircle and the other two are tangent to each other at a vertex. To determine which circle must be a circumcircle we proceed as follows. Consider the point A_3. The circle on A_1PA_2 will be a circumcircle if

$$\angle\,(\tfrac{2}{3}\pi - kA_3') = \angle\,(A_3 + x\pi) \quad (x \text{ an integer}).$$

Draw from O_1 the center of the circle through A_2PA_3 a line to A_3. Then

$$\angle\, O_1A_3A_2 = \angle\,(\tfrac{1}{6}\pi - kA_1').$$

Similarly for the circle on A_3PA_1 we have

$$\angle\, O_2A_3A_1 = \angle\,(\tfrac{1}{6}\pi - kA_2').$$

Also

$$\begin{aligned}\angle O_1A_3A_2 + \angle O_2A_3A_1 + \angle A_3 &= \angle\,(\tfrac{1}{3}\pi - k(A_1' + A_2')) + \angle\, A_3.\\ &= \angle\,(\tfrac{1}{3}\pi + kA_3' + A_3) (\text{since } A_1' + A_2' + A_3' = 0)\\ &= \angle\,[\pi - (\tfrac{2}{3}\pi - kA_3') + A_3]\\ &= \angle\,(\pi - A_3 - x\pi + A_3\\ &= \angle\,(1 - x)\pi.\end{aligned}$$

Therefore angle $O_1A_3O_2$ is a straight angle. It is evident that the point A_3 will not be a point of the curve C unless these conditions hold. These facts give the following theorem:

A vertex of the triangle will be a point of the curve C, when and only when the circle on the opposite side as segment, becomes a circumcircle.

Let the differences A_n' be commensurable. We then have $|A_1'|:|A_2'|:|A_3'| = p_1:p_2:p_3$, where the p_i are relatively prime integers. Consider again the point A_3. It will be a point of the curve for more than one value of k. Let k_1 and k_2 be two consecutive values of k which make P_3 a point of C. Then $(k_2-k_1)A_3' = \pi$. But while this change takes place, the angle $O_2A_2A_1$ changes $(k_2-k_1)A_2' = (p_2/p_3)\pi$. And since p_2 and p_3 are relatively prime it will take p_3 rotations to bring the line O_2A_3 to its original position. Moreover all such rotations will be equal, and the construction shows that the tangents to C at A_3 will be perpendicular to O_2A_3. Hence the theorem:

If the angles of a triangle differ from $\pi/3$ in the ratio $p_1:p_2:p_3$ and the p's are relatively prime integers, the curve C has a multiple point of order p_i at A_i and the tangents at these multiple points divide the angular magnitude into $2p_i$ equal parts.

If there are other multiple points on C, the centers of the circles determining them will be identical for each branch of C through the point and will approach their respective positions always from the same direction. Hence there can be no other multiple points on C.

The coordinates of the point P may be written:

$$\alpha_i = M\frac{\sin\,(\tfrac{2}{3}\pi - kA_i')}{\sin\,(\tfrac{1}{3}\pi - (k-1)A_i')} \tag{4}$$

where $M = f(k)$.

If the A_i' are commensurable the equation of the curve will be algebraic. Its equation may be obtained as follows.

Equation (4) may be written:

$$\tan kA_i' = \frac{\alpha_i \sin(\tfrac{1}{3}\pi + A_i') - M \sin \tfrac{1}{3}\pi}{\alpha_i \cos(\tfrac{1}{3}\pi + A_i') - M \cos \tfrac{1}{3}\pi}, \tag{5}$$

Let $A_1' = p_1\theta$, $A_2' = p_2\theta$, $A_3' = -p_3\theta$. Then since

$$\tan(p_2 k A_1') = \tan(p_1 k A_2'), \tag{6}$$

a substitution from (5) in (6) gives

$$\begin{aligned} &\frac{\alpha_1^{p_2}\sin\left(\frac{p_2\pi}{3} + p_1p_2\theta\right) + p_2\alpha_1^{p_2-1}M\sin\left[(p_2-2)\frac{\pi}{3} + (p_1p_2 - p_1)\theta\right] + \cdots + M^{p_2}\sin - \frac{p_2\pi}{3}}{\alpha_1^{p_2}\cos\left(\frac{p_2\pi}{3} + p_1p_2\theta\right) + p_2\alpha_1^{p_2-1}M\cos\left[(p_2-2)\frac{\pi}{3} + (p_1p_2 - p_1)\theta\right] + \cdots + M^{p_2}\cos - \frac{p_2\pi}{3}}, \\ &= \frac{\alpha_2^{p_1}\sin\left(\frac{p_1\pi}{3} + p_1p_2\theta\right) + p_1\alpha_2^{p_1-1}M\sin\left[(p_1-2)\frac{\pi}{3} + (p_1p_2 - p_2)\theta\right] + \cdots + M^{p_1}\sin - \frac{p_1\pi}{3}}{\alpha_2^{p_1}\cos\left(\frac{p_1\pi}{3} + p_1p_1\theta\right) + p_1\alpha_2^{p_1-1}M\cos\left[(p_1-2)\frac{\pi}{3} + (p_1p_2 - p_2)\theta\right] + \cdots + M^{p_1}\cos - \frac{p_1\pi}{3}}, \end{aligned} \tag{7}$$

an equation of degree p_1+p_2 in M.

Moreover $\tan(kA_1' + kA_2') = -\tan kA_3'$, and this equation by virtue of (5) simplifies to

$$M = \frac{\alpha_2\alpha_3 \sin A_1' + \alpha_1\alpha_3 \sin A_2' + \alpha_1\alpha_2 \sin A_3'}{\alpha_1 \sin A_1' + \alpha_2 \sin A_2' + \alpha_3 \sin A_3'}. \tag{8}$$

The value of M in (8) when substituted in (7) gives a curve of degree $p_1+p_2+p_3$ in the α's. We therefore have the following theorem.

If p_1, p_2 and p_3 are relatively prime integers the curve C is algebraic and of degree $p_1+p_2+p_3$.

Fig. 3 shows the curve for $p_1=1$, $p_2=2$ and $p_3=3$.

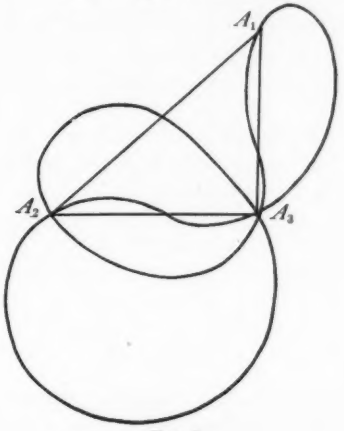

FIG. 3

MEAN VALUE OF THE ORDINATE OF THE LOCUS OF THE RATIONAL INTEGRAL ALGEBRAIC FUNCTION OF DEGREE n EXPRESSED AS A WEIGHTED MEAN OF $n+1$ ORDINATES AND THE RESULTING RULES OF QUADRATURE

By BENJAMIN F. GROAT, Brookline, Mass.

From time to time there have appeared summaries of such rules as Newton's, Simpson's (so called, but due to Cotes), and others, for "approximating" the areas of inclosures, and the subject of interpolation has received attention by Newton, Lagrange, and Gauss. But the author is not aware of any general formulas for finding the averaging weights of arbitrarily chosen ordinates, such as will give the true area under the general curve of rational integral algebraic form. Nor does he know of an instance in which a rule has been given for finding the mean ordinate of such a curve as the arithmetical mean of a determinate number of properly situated ordinates.

The following treatise produces the mean ordinate under a given segment of the general curve as a weighted mean, along with associated general formulas for the weights of the ordinates. All the usual special rules mentioned, and doubtless others, may be derived immediately from these formulas as special cases, while any new arbitrary disposition of ordinates may be analyzed. It is urged, in particular, that the rules herein formulated for the arithmetical mean are generally the most direct and most easily applied, especially those for which the ordinates are equally spaced. There is an infinite number of ordinate dispositions each furnishing the arithmetical mean.

Let the general integral algebraic curve be

$$y = a\xi^{n-1} + b\xi^{n-2} + \cdots + \epsilon, \tag{1}$$

n being the number of ordinates to be employed with it. The base above which the area is to be found is represented by D. As the results are independent of the coordinate axes, the origin may be at one extremity of the base. The coordinates of the heads of the ordinates are, respectively, $(\xi_1 y_1), \cdots, (\xi_n y_n)$. The mean value of the ordinate over the base, D, is represented by V, such that the area is $A = VD$.

The essence of the problem consists in finding V as an arbitrarily weighted mean of n ordinates, or, in terms of parameters, the n parameters, $a, \cdots, \epsilon$, are to be eliminated from the expression for the mean value of the ordinate, while the n ordinates, $y_1, \cdots, y_n$, erected above the base, are to be introduced in their stead.

The mean value of the ordinate of A is

$$V = \frac{A}{D} = \frac{aD^{n-1}}{n} + \frac{bD^{n-2}}{n-1} + \cdots + \epsilon. \tag{2}$$

The difference between the mean value, (2), of the ordinate, and the value, (1), of any particular ordinate may be found by subtraction. Then, since there

may be calculated (or measured) n ordinates at arbitrarily chosen distances from the origin, there may be formed n equations of the type

$$(3) \qquad (V - y_z) + aD^{n-1}\left(x_z{}^{n-1} - \frac{1}{n}\right) + bD^{n-2}\left(x_z{}^{n-2} - \frac{1}{n-1}\right) + \cdots = 0,$$

in which the relative abscissa, x, is written for $\xi \div D$, while z may be any integer from 1 to n.

From these n equations the remaining $n-1$ parameters, $a, b, \cdots$, may be eliminated, ϵ having been eliminated by the subtractions. The eliminating determinant is

$$(4) \qquad \left[(V - y_1), \left(x_2^{n-1} - \frac{1}{n}\right), \left(x_3^{n-2} - \frac{1}{n-1}\right), \cdots, \left(x_n - \frac{1}{2}\right)\right]$$

which, placed equal to zero and solved for V, yields

$$(5) \qquad V = \frac{\sum y_z C_z}{\sum C_z},$$

wherein C_z is the cofactor of y_z in the determinant (4).

Thus the mean ordinate is a weighted mean of the n calculated (measured in practical mensuration) ordinates. The weights depend entirely upon the relative distances of the ordinates from the origin, or conversely. In particular, the weight of y_m, in determinant form, is

$$(6) \quad W_m = C_m = (-1)^{m+1}\left[\left(x_1^{n-1} - \frac{1}{n}\right), \left(x_2^{n-2} - \frac{1}{n-1}\right), \cdots, \left(x_n - \frac{1}{2}\right)\right]$$

from which the constituents containing the symbols $x_m{}^z$ must be omitted.

This determinant clearly breaks up into 2^{n-1} determinants,all but n of which vanish, thus leaving

$$(7) \quad W_m = (-1)^{m+n}\left\{\frac{1}{n}[x_1^{n-2}, x_2^{n-3}, \cdots, x_n, 1] - \frac{1}{n-1}[x_1^{n-1}, x_2^{n-3}, \cdots, x_n, 1]\right.$$
$$\left. + \cdots + (-1)^{n-1}[x_1^{n-1}, x_2^{n-2}, \cdots, x_n]\right\},$$

in which terms, excepting the last, the successive determinants, each containing $n-2$ exponents, are written out by omitting, progressively, one at a time, every exponent, beginning with the first, but including all in the last term. This is the general formula for the weight of an ordinate.

The series (7) of determinants is divisible by the determinant

$$(8) \qquad [x_1^{n-2}, x_2^{n-3}, \cdots, x_n, 1],$$

which is merely the product of all the binomial differences, $(n-1)(n-2) \div 2$ in number, (subscript m not appearing) which can be formed by taking all combinations of the constituent ordinates two at a time. If the mode of forming factors be: x_1-x_2, x_1-x_3, $\cdots$, x_2-x_3, x_2-x_4, $\cdots$, etc., following the order of the principal diagonal, the product is treated with the positive sign.

Therefore the relative weight C_m may be written

$$(9)\quad W_m = (-1)^{m+n}P_m\left\{\frac{1}{n} - \frac{\sum x}{n-1} + \frac{\sum xx}{n-2} - + \cdots + (-1)^{n-1}(xxx\cdots x)\right\}$$

in which only $n-1$ relative abscissas appear, the abscissa subscript, m, being omitted on the right, and wherein

$$(10)\qquad P_m = (x_1 - x_2)(x_1 - x_3)\cdots(x_2 - x_3)(x_2 - x_4)\cdots(x_{n-1} - x_n),$$

the factors containing the subscript m being omitted in accord with (9).

Equation (9) is a convenient expression of the general formula (7) for the weight of any ordinate corresponding with the particular choice of the numerical value of m. By it, or any one of the formulas for W_m, one may deduce many of the rules of mensuration as special cases:

$n=2$. The curve is $y=a\xi+b$, a straight line. If $x_1=0$, $x_2=1$, i.e., if $\xi_1=0$, and $\xi_2=D$, in (1), then, by (9) the relative weights of y_1 and y_2 become equal, which is the well-known trapezoidal rule of mensuration, the mean ordinate being arithmetical mean of the two ordinates.

$n=3$. $x_1=0$, $x_2=\frac{1}{2}$, $x_3=1$. Then, by (9), the relative weights are as 1:4:1, "Simpson's rule," frequently employed in engineering to calculate approximate areas of irregular inclosures.

$n=4$. $x_1=0$, $x_2=\frac{1}{3}$, $x_3=\frac{2}{3}$, $x_4=1$. Then, by (9), $W_1:W_2:W_3:W_4=1:3:3:1$, Newton's rule, also employed to calculate areas.

$n=5$. $x_1=0$, $x_2=\frac{1}{4}$, $x_3=\frac{1}{2}$, $x_4=\frac{3}{4}$, $x_5=1$. By (9) the relative weights are, respectively, 7, 32, 12, 32, 7, which is a rule given in the 14th edition of the Encyclopaedia Britannica in an article on mensuration by W. F. Sheppard.

$n=6$. $x_1=0$, $x_6=1$, the base being divided into equal parts by equally-spaced ordinates as before. By (9) the weights are, respectively, 19, 75, 50, 50, 75, 19. This is probably a new rule altogether.

This method may be continued indefinitely.

Let attention now revert to $n=3$, but, instead of locating an ordinate at each end of the base, as has been customary, let the abscissas x and y (y may here represent a particular one of the abscissas for simplicity of symbols. y_z would be an ordinate.) be made arbitrary, but equally distant from the ordinate z in the central position. Then, by (9), since $z=\frac{1}{2}$ and $x+y=1$, the relative weights become

$$(11)\quad W_x = \frac{1}{24}(2x-1),\quad W_z = (2x-1)\left(x^2 - x + \frac{1}{6}\right),\quad W_y = \frac{1}{24}(2x-1)$$

when expressed in terms of x.

Equations (11) illustrate the general theorem that, in any system of ordinates symmetrically disposed as to the center of the base, the weights of any two ordinates equally distant from that center are equal, while the weight of the central ordinate, if there is one, depends upon the displacements of the others. Hence, if W_z be made to vanish, the mean ordinate of the area sought will be the arithmetical mean of the ordinates y_x and y_y. Thus, rejecting $x=\frac{1}{2}$,

$$x = \tfrac{1}{2} \pm \sqrt{(1/12)}, \tag{12}$$

while y is the complementary surd. This special case of the general formulas is well known to hydraulic engineers as the "two-tenths eight-tenths depth method" for making observations by current meters of velocities of flowing water at 0.2 and 0.8 the full depth of the stream, although the present analysis shows the terminology to be inexact.

This result shows, not only that the areas of all the parabolas passing through the two fixed points are equal, but, also, that this fixed area is equal to the area under the straight line through the two points as limited by the common base. Passing on to a cubic with four ordinates symmetrically placed as to center of base, it will be found, by making the weight of each of one pair of ordinates equal to zero, that the abscissas of the other pair must equal x in (12). Thus, the area under the cubic is equal to the area under the straight line, and also to a fixed area under a family of parabolas. Todhunter has called attention to the fixity of such an area under a variable curve of odd degree, but it does not seem that he called attention to the resulting equality of such areas for curves of lower degree, nor to the fact that the curve of highest degree may be of even degree (Todhunter's *Treatise on Integral Calculus*, p. 398, Art. 400).

Instead of making $W_z=0$, suppose it be demanded that all three weights shall be the same, z still being the central abscissa. It is merely necessary to equate W_z with either of the other weights, since, by the theorem just given, the latter are equal. Thus one easily finds

$$x = \tfrac{1}{2} - \sqrt{\tfrac{1}{8}},\ z = \tfrac{1}{2},\ y = \tfrac{1}{2} + \sqrt{\tfrac{1}{8}}. \tag{13}$$

This result gives the theorem: *When three velocity readings are taken respectively at* $\frac{1}{2}-\sqrt{\frac{1}{8}}$, $\frac{1}{2}$, *and* $\frac{1}{2}+\sqrt{\frac{1}{8}}$, *relative depth in the vertical velocity plane, the mean velocity in that plane is the arithmetical mean of the three observed velocities, or,*

$$V = (y_x + y_z + y_y) \div 3, \tag{14}$$

assuming that the velocity curve may be closely approximated by a parabola with horizontal axis, which is known to be the case. Of course the method applies to finding areas (also volumes) just as with Simpson's and Newton's rules. The theorem is exact for any parabolic curve or for a straight line. It has probably not been given before.

But the sole restriction, $W_x=W_z=W_y$, is only partial, for it amounts to two equations involving three unknowns, thus leaving two variables dependent upon

the third. Hence, with x, y, z, undetermined, and by eliminating z, it may be shown that

$$(15)\quad (y-x)^2(y+x-2xy-\tfrac{2}{3})^2\{4(x^2+y^2-1)+[3-2(x+y)^2]\}=0,$$

when all three weights are the same. Consequently there is an infinite number of arrangements of three ordinates the arithmetical mean of which is the mean ordinate under a parabolic segment, as is otherwise very apparent.

After rejecting inapplicable solutions the values of y and z in terms of x may be shown to be

$$(16)\quad \begin{aligned} y &= \frac{3}{4}-\frac{x}{2}\pm\left[\frac{3}{4}(x-x^2)-\frac{1}{16}\right]^{1/2} \\ z &= \frac{-(x^2-\tfrac{1}{3})\pm[(x^2-\tfrac{1}{3})^2+(1-2x)G]^{1/2}}{1-2x}, \end{aligned}$$

where

$$G=x^2-\frac{2}{3}x\pm\left\{\frac{5}{24}-(x-x^2)\right\}[12(x-x^2)-1]^{1/2}.$$

This completely evaluates the problem for three ordinates. It is now merely necessary to ascertain what values may be assigned to the arbitrary variable so as to determine useful sets of values of x, y, z, the relative abscissas. Some of the results will be found in Table 1. They may be checked by means of the preceding formulas. The weights for other arrangements of ordinates may be calculated and appended as desired; or, ordinate dispositions may be determined for any assigned weights.

$n=4$. *New formulas.* The usual mensurational arrangement of ordinates for this case has previously been shown to be a special case easily soluble by (9). Further development is not only possible, but leads to that most facile form, the arithmetical mean. The conditions for this latter case are: $W_w=W_x=W_y=W_z$, although only one equation is available on account of the theorem of symmetry previously stated, which is here to be applied:

$$w+z=x+y=1 \text{ and } w-x=x-y=y-z.$$

An appeal to (9) brings forth

$$(17)\qquad w=\tfrac{1}{2},\ w=\tfrac{1}{2}-\sqrt{(0.15)} \text{ and } V=\tfrac{1}{4}[y_w+y_x+y_y+y_z],$$

in which z takes the complementary value of w, while the value $\frac{1}{2}$ for w is rejected. The values of the other abscissas and the results of other solutions appear in Table 1.

$n=5$. *New formulas.* The usual case of mensuration has been treated previously. Two other cases, however, appear in Table 1. They are: equally spaced ordinates symmetrically situated as to the center, which does not admit

the arithmetical mean; weights equal so as to admit the arithmetical mean, which case does not admit equal spacings when one ordinate is central and the others symmetrically situated as to it.

$n=6$. *New formulas.* Both formulas herein developed are new. The case corresponding to those of mensuration has already been treated.

$n=7$. *New formulas.* The base is divided into 6 equal parts. There are seven ordinates; one at each point of division, including the two ends of base. Weights are, respectively, 41, 216, 27, 272, 27, 216, 41, total, 840.

Weddle's rule gives weights in the order: 1, 5, 1, 6, 1, 5, 1, but it is not exact for seven ordinates to a curve of the sixth degree.

Ordinates beyond the limits of the base may have negative weights, as in the following cases of five ordinates: weights: 29, 124, 24, 4, -1; abscissas: 0, $\frac{1}{2}$,

TABLE 1

A few special quadrature rules derived from the general rule (9).

Three Ordinates

Locations R_x	R_y	R_z	*Weights* W_x	W_y	W_z	*Ordinate Spacing*	*Remarks*
0	$\frac{1}{2}$	1	1	4	1	$\frac{1}{2}$	Simpson's rule (Cotes)
.1	.5	.9	100	184	100	.4	
1/8	1/2	7/8	24	33	24	3/8	
1/7	1/2	6/7	49	52	49	5/14	
1/6	1/2	5/6	3	2	3	1/3	
1/6	1/3	5/6	3	4	5	(1/6−(3/6)	
1/5	2/5	4/5	16	3	17	(1/5)−(2/5)	
$\frac{1}{2}-r$	$\frac{1}{2}$	$\frac{1}{2}+r$	1	0	1	r	$r=\sqrt{(1/12)}$
.1	$.7-r$	$.7+r$	1	1	1	$(.6-r):2r$	$r=\sqrt{(1/200)}$ Arith. mean
$\frac{1}{2}-r$	$\frac{1}{2}$	$\frac{1}{2}+r$	1	1	1	r	$r=\sqrt{(1/8)}$ " "
1/6	$\frac{2}{3}-r$	$\frac{2}{3}+r$	1	1	1	$(\frac{1}{2}-r):2r$	$r=\sqrt{(1/24)}$ " "

Four ordinates—Symmetry as to center

R_w, R_z	R_x, R_y	W_w, W_z	W_x, W_y		
$\frac{1}{2}\mp\frac{1}{2}$	$\frac{1}{2}\mp\frac{1}{2}$	1	3	1/3	Newton's rule 1:3:3:1
$\frac{1}{2}\mp r$	$\frac{1}{2}\mp\frac{1}{3}r$	1	1	$2r/3$	$r=\sqrt{.15}$ Arith. mean (New)

Five ordinates—Symmetry as to center

R_v, R_z	R_w, R_y	R_x	W_v, W_z	W_w, W_y	W_x		
$\frac{1}{2}\mp\frac{1}{2}$	$\frac{1}{2}\mp\frac{1}{4}$	$\frac{1}{2}$	7	32	12	1/4	†
$\frac{1}{2}\mp r$	$\frac{1}{2}\mp\frac{1}{2}r$	$\frac{1}{2}$	17	17	22	$\frac{1}{2}r$	$\sqrt{(3/17)}$ Probably new.
$\frac{1}{2}\mp r_1$	$\frac{1}{2}\mp r_2$	$\frac{1}{2}$	1	1	1	$(r_1-r_2):r_2$	‡

* The "two-tenths eight-tenths depth method" applied to velocity measurements by the current meter.

† Given by Sheppard in the article on *Mensuration* in the Encyclopedia Brittaninica, 14th edition

‡ $r_1=\left[\frac{1}{4}-\frac{7-\sqrt{11}}{48}\right]^{1/2}$; $r_2=\left[\frac{1}{4}-\frac{7+\sqrt{11}}{48}\right]^{1/2}$. Arithmetic mean. New.

1, 3/2, 2, and weights: $-1, 34, 114, 34, -1$; abscissas: $-\frac{1}{2}, 0, \frac{1}{2}, 1, 3/2$. All the ordinates may be beyond the limits of the base.

Forms (9) and (10) are general, and convenient for a small number of ordinates. In cases where the ordinates are symmetrically arranged as to a point on the base, or its extension, the product sums of (9) and the product (10)may be simplified when there are too many ordinates for convenient use of (9). Thus, spacings being symmetrically placed about a given point of abscissa, y, on the line of base, the following are examples of forms for total summations, including all the abscissas:

$$\sum x = ny; \quad \sum xx = \sum_1^{\frac{1}{2}(n-k)} a_z + \frac{(n+k-2)(n-k)}{2} y^2;$$

$$\sum xxx = (n-2)y \left[\sum_1^{\frac{1}{2}(n-k)} a_z + \frac{n(n-4)+3k}{6} y^2 \right]; \tag{18}$$

where $k = \frac{1}{2}[1 - (-1)^n]$, $a_z = x_z(2y - x_z)$, and y is the abscissa of the point of symmetry on the base, or its extension, and the unlimited summations include the n abscissas.

The forms (9) and (10) do not include x_m, but they may be expressed in terms of (18) and x_m as follows:

$$\sum_m x = ny - x_m; \sum_m xx = \sum_1^{\frac{1}{2}(n-k)} a_z - a_m - (n-2)yx_m + \frac{(n-k)(n+k-2)}{2} y^2; \tag{19}$$

$$\sum_m xxx = [(n-2)y - x_m]\left[\sum_1^{\frac{1}{2}(n-k)} a_z - a_m\right] + (n-2)\frac{n(n-4)+3k}{6} y^3 - \frac{(n-2)(n-k-4)+k(n-k)}{2} y^2 x_m.$$

These forms result from successive applications of

$$\sum(x \cdots x_r) = \sum_m x \sum_m (x \cdots x_{r-1}) \div r,$$

$$\sum_m (x \cdots x_r) = \sum(x \cdots x_r) - x_m \sum_m (x \cdots x_{r-1}) \tag{20}$$

the last term of the first of which is exactly divisible by r. The subscript, m, of $\sum_m$, indicates omission of x_m. The number of factors in a product is shown by the subscript r. When n is odd, the odd ordinate has the abscissa, y, but the formulas may be arranged otherwise, if needed. If P_m is the product of all binomial differences of the n abscissas taken as shown in (10), omitting factors containing x_m, then

$$P_m = \frac{(n-1)_{m-1}(n-1)!!}{(m-1)!(n-1)!}\left(\frac{x_n - x_1}{n-1}\right)^{\frac{1}{2}(n-1)(n-2)} \tag{21}$$

wherein

$$(n-1)_{m-1} = (n-1)(n-2)\cdots(n-m+1)$$

and $(n-1)!!=(n-1)!\,(n-2)!\cdots 2!\,1$. Thus the relative values of P_1, P_2, etc., follow Newton's rule for binomial coefficients, when ordinate spacings are equal. (For five ordinates, $P_1:P_2:P_3=1:4:6$ etc.).

In solutions of complicated cases factors of the forms $2u-1$, $2v-1$, $u+z-1$, may be cancelled from equations expressing equal weights, and the degree of such an equation may be depressed by placing $a=u-u^2$, $b=v-v^2$, etc. For example, with $n=6$, the quantities to be equated for equal weights reduce to $(-1)^{m+n}\ F[1-5(b+c)+30bc]$; forms deducible from one another by cyclic changes, F being a factor dependent upon m. $F=1, 3, 2$, respectively, when spacings are equal and $m=1, 2, 3$.

A SIMPLE DERIVATION OF WARING'S FORMULAE

By FRANCIS D. MURNAGHAN, Johns Hopkins University

The formulae giving the expressions for the elementary symmetric functions of the roots of an algebraic equation in terms of the various sums of like powers of the same roots were published by Waring in his *Miscellanea analytica* in 1762. It is hoped that the following derivation of these expressions may prove of interest.

A square matrix $\boldsymbol{A}$ with elements a_s^r, may be regarded as a linear vector function [=mixed tensor of rank 2] which transforms a vector $x=(x^1, x^2, \cdots, x^n)$ into a vector y by means of the equality $\boldsymbol{A}\,x=y$ which may be written out explicitly as

$$a_\alpha{}^r x^\alpha = y^r; \qquad (r = 1, 2, \cdots n;\ \alpha \text{ a summation symbol}).$$

The proper or characteristic numbers of the matrix are those numbers λ for which $y=\lambda x$ so that $\boldsymbol{A}x=\lambda x$. On transforming this vector equality by means of the matrix $\boldsymbol{A}$ we obtain

$$\boldsymbol{A}\boldsymbol{A}x = \boldsymbol{A}\lambda x = \lambda \boldsymbol{A}x = \lambda^2 x,$$

so that[1] the proper numbers of $\boldsymbol{A}\boldsymbol{A}(\equiv \boldsymbol{A}^2)$ are the squares of the proper numbers of $\boldsymbol{A}$. A continuation of this procedure shows at once, that the proper numbers of $\boldsymbol{A}^p$, where p is any positive integer, are the p^{th} powers of the proper numbers of $\boldsymbol{A}$. Here $\boldsymbol{A}^p$ is the matrix of which the element in the r^{th} row and s^{th} column is

$$a^r_{\alpha_1} a^{\alpha_1}_{\alpha_2} \cdots a^{\alpha_{p-1}}_{s};\ \alpha_1, \cdots, \alpha_{p-1} \text{ being summation symbols.}$$

[1] See Aurel Wintner, *Spektraltheorie der unendlichen Matrizen* (1929), p. 15.

Now the proper numbers of a matrix are the roots of the *proper equation* which is obtained by equating to zero the determinant of the matrix $A-\lambda E$ whose r, s element is $a_s^r-\lambda\delta_s^r$; $\delta_s^r=0$ if $r\neq s$ and $=1$ if $r=s$. Hence the sum of the proper numbers of the matrix A is a_α^α. More generally.

$$\sum\lambda_1 = a_\alpha^\alpha = \delta_\beta^\alpha a_\alpha^\beta,$$

$$\sum\lambda_1\lambda_2 = \frac{1}{2!}\delta_{\beta_1\beta_2}^{\alpha_1\alpha_2} a_{\alpha_1}^{\alpha_1} a_{\alpha_2}^{\beta_2},$$

. . . . . .

. . . . . .

$$\sum\lambda_1\lambda_2\cdots\lambda_r = \frac{1}{r!}\delta_{\beta_1\beta_2\cdots\beta_r}^{\alpha_1\alpha_2\cdots\alpha_r} a_{\alpha_1}^{\beta_1}\cdots a_{\alpha_r}^{\beta_r},$$

where

$$\delta_{n_1\ \cdots\ n_r}^{m_1\ \cdots\ m_r}$$

is the generalized Kronecker delta[2] and r runs from 1 to n.

We shall denote by S_r the r^{th} elementary symmetric function of the proper numbers λ of A and by P_r the sum of the r^{th} powers of these numbers so that (since P_r is the first elementary symmetric function of the proper numbers of A^r)

$$P_r = a_{\alpha_1}^{\alpha_r} a_{\alpha_2}^{\alpha_1}\cdots a_{\alpha_r}^{\alpha_{r-1}}.$$

The numerical coefficient

$$\delta_{\beta_1\beta_2\cdots\beta_r}^{\alpha_1\alpha_2\cdots\alpha_r}$$

in the expression

$$r!S_r = \delta_{\beta_1\cdots\beta_r}^{\alpha_1\cdots\alpha_r} a_{\alpha_1}^{\beta_1}\cdots a_{\alpha_r}^{\beta_r},$$

being a determinant of which the element in the r^{th} row and s^{th} column is

$$\delta_{\beta_s}^{\alpha_r},$$

may be expanded as a sum of terms of the type

$$\pm\ \delta_{\gamma_1}^{\alpha_1}\cdots\delta_{\gamma_r}^{\alpha_r}$$

where $(\gamma_1, \cdots, \gamma_r)$ is a permutation of the numbers $(\beta_1, \cdots \beta_r)$. If this permutation is the identity (or, in other words, consists of r cycles of one letter each) the corresponding term in S_r is

$$\delta_{\beta_1}^{\alpha_1}\ \delta_{\beta_2}^{\alpha_2}\cdots\delta_{\beta_r}^{\alpha_r} a_{\alpha_1}^{\beta_1}\cdots a_{\alpha_r}^{\beta_r}$$

[2] See *The generalized Kronecker symbol and its application to the theory of determinants*, This Monthly, vol. 32 (1925), pp. 233–241; O. Veblen, *Invariants of quadratic differential forms*, p. 3.

On summing with respect to the summation labels $(\beta_1, \cdots \beta_r)$ this becomes

$$a_{\alpha_1}^{\alpha_1} \cdots a_{\alpha_r}^{\alpha_r},$$

and on further summation with respect to the labels $(\alpha_1, \cdots \alpha_r)$ we get P_1^r. If, again, the permutation $(\gamma_1, \cdots \gamma_r)$ of the labels $(\beta_1, \cdots, \beta_r)$ has one cycle of two letters and $r-2$ cycles of one letter [e.g., $(\gamma_1, \gamma_2, \gamma_3 \cdots, \gamma_r) = (\beta_2, \beta_1, \beta_3, \cdots, \beta_r)$] the corresponding term in S_r is

$$-\delta_{\beta_2}^{\alpha_1}\delta_{\beta_1}^{\alpha_2}\delta_{\beta_3}^{\alpha_3} \cdots \delta_{\beta_r}^{\alpha_r} a_{\alpha_1}^{\beta_1} \cdots a_{\alpha_r}^{\beta_r}$$

and summation with respect to the labels β gives

$$-a_{\alpha_1}^{\alpha_1} a_{\alpha_2}^{\alpha_1} a_{\alpha_3}^{\alpha_3} \cdots a_{\alpha_r}^{\alpha_r}.$$

Further summation with respect to the labels α yields $-P_2P_1^{r-2}$. Now the number of permutations of r letters which are mere transpositions is

$$\binom{r}{2} = \frac{r(r-1)}{1 \cdot 2}.$$

Hence there will be $(r/2)$ terms of the type just discussed and these will yield, on summation,

$$-\binom{r}{2} P_2 P_1^{r-2}.$$

The general permutation of the r letters $(\beta_1, \cdots \beta_r)$ may be analysed into m_1 cycles of one letter, m_2 cycles of 2 letters, m_3 cycles of 3 letters, $\cdots m_r$ cycles of r letters where

$$m_1 + 2m_2 + 3m_3 + \cdots + rm_r = r$$

and a term in S_r corresponding to such a permutation $(\gamma_1, \gamma_2, \cdots, \gamma_r)$ of the r letters $(\beta_1, \cdots, \beta_r)$ will yield on summation with respect to the labels β and α the result

$$(-1)^{m_2+m_4+m_6+\cdots} P_1^{m_1} P_2^{m_2} \cdots P_r^{m_r}.$$

The number N of permutations on r letters which consist of exactly m_1 cycles of one letter, m_2 cycles of 2 letters, $\cdots m_r$ cycles of r letters, was calculated by Cauchy[1] and the derivation is extremely simple although the result does not, as far as we are aware, find its place in the text books on algebra. Since the order in which the cycles are given does not affect the permutation and since the particular letter of a given cycle with which we choose to start the cycle does not affect the permutation we obtain all the permutations on the r letters by multiplying N by $m_1!\, m_2! \cdots m_r!\, 2^{m_2} 3^{m_3} \cdots r^{m_r}$. Hence

[1] Cauchy, Oeuvres, Ser. 1, Tome 9, p. 289.

$$N = \frac{r!}{m_1!m_2! \cdots m_r! 2^{m_2} \cdot 3^{m_3} \cdots r^{m_r}}$$

so that, finally,

$$(W) \qquad S_r = \sum (-1)^{m_2+m_4+m_6\cdots} \frac{P_1^{m_1} P_2^{m_2} \cdots P_r^{m_r}}{m_1!m_2! \cdots m_r! 2^{m_2} 3^{m_3} \cdots r^{m_r}}$$

the summation being over all positive integral solutions of the equation

$$m_1 + 2m_2 + \cdots + rm_r = r.$$

This is the statement of Waring's formula. In our derivation we started with the matrix A and the $(\lambda_1, \cdots, \lambda_n)$, its proper numbers, were secondary. However it is quite evident that if the $(\lambda_1, \cdots \lambda_n)$ are given arbitrarily we need merely choose as our matrix A the diagonal matrix $a_s^r = 0$, if $r \neq s$, $a_r^r = \lambda_r$ in order to have, as its proper numbers, the arbitrary numbers $(\lambda_1, \cdots \lambda_n)$. Hence the proof given is valid for any algebraic equation. The main interest in the present derivation is the identification of the coefficients in the formula (W) with the number of permutations on r letters whose cycle structure is given.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A SIMPLE GEOMETRICAL PARADOX

By J. L. COOLIDGE, Harvard University

Question: A point P in three-dimensional Euclidean space is required to be collinear with two given points A and B; how many independent algebraic conditions are thereby imposed upon its coordinates?

First answer: In order to be collinear with A and B it is necessary and sufficient that P should be coplanar with A and B and each of two arbitrary points not in the same plane through A and B; *two conditions.*

Second answer: If P be collinear with A and B it is necessary and sufficient that the sum of two of the distances determined by two pairs from these three points should be equal to the distance determined by the third pair. That may be written

$$(-AB + BP + PA)(AB - BP + PA)(AB + BP - PA) = 0.$$

We rationalize the left hand side by multiplying through by a non-vanishing factor:

$$-(AB+BP+PA)(-AB+BP+PA)(AB-BP+PA)(AB+BP-PA)=0.$$

This expands into

$$AB^4 + BP^4 + PA^4 - 2BP^2 \cdot PA^2 - 2PA^2 \cdot AB^2 - 2AB^2 \cdot BP^2 = 0.$$

The left hand side is an unfactorable rational expression in the coordinates of the point P, hence *but one condition is imposed.*

Which answer is right? How many conditions are imposed in a Euclidean space of n dimensions?

Prize Problems

By Tomlinson Fort, Lehigh University

Professor Radò in his lecture on "Mathematics in Hungary," delivered before the Mathematical Association of America at its meeting in Cleveland, Ohio (Jan. 1, 1931), described certain prizes given in Hungary for the solution of problems in elementary mathematics.

Professor Radò's remarks greatly interested the persons present. It consequently seems to me that the readers of The Monthly will be interested in learning of a series of prize problems which has been in existence at Lehigh University for the past three years. Each semester five problems are published in the college paper, one each two weeks, over a ten-week period. Problems are published on Fridays. Solutions must be handed in not later than the next subsequent Wednesday. A prize is offered for the best solution of the problem by a Lehigh undergraduate, and, in addition, the name of the winner is posted on our bulletin board and published in the college paper. At first we made each prize five dollars, but subsequently changed it, making each prize a mathematical book to be properly inscribed and not to exceed five dollars in value.

We consider the prize problems a success. There has been so much favorable comment that the President of the University, who was at first very dubious, now no longer questions the inclusion of this item in our budget. We have a departmental committee to choose problems, read papers, and handle other details. It has proved harder than we anticipated to originate problems of interest and of proper difficulty. However, I think the set of problems finally evolved a good one. Decision between papers has at times necessarily been made on minor points, and in a few instances prizes have been divided.

The prize problem series at Lehigh was suggested to me some years ago by a conversation with Professor Lester S. Hill, who told me of something of the kind that had been tried at one time at Yale University.

On Division with a Calculating Machine

By E. C. Kennedy, University of Texas, College of Mines

An eight-column calculating machine may be used to divide a number of sixteen (or less) digits by a number of sixteen digits in the following manner:

Let the quotient be N/n and let n_1 be the number consisting of the first eight digits of n.

Set $$(N/n) = (N/n_1) + e,$$

where N/n_1 is taken to eight places and e is to be determined.

Then

$$n[(N/n_1) + e] = N$$

and

$$e = \frac{N - n(N/n_1)}{n_1}, \text{ approximately.}$$

Then the quotient $= (N/n_1) + e$.

This does not require division by a number of more than eight digits and requires only one such multiplication.

For example, evaluate .423456/.6823499215826795 to sixteen places.

Here

$$N/n_1 = .62058481 \text{ (to eight places only);}$$

and

$$N - n(N/n_1) = .0000000035610979$$

and

$$e = .0000000035610979/.68234992 = .0000000052188735.$$

Therefore the quotient $= .6205848152188735$ within an error of about one or less in the sixteenth place.

RECENT PUBLICATIONS

EDITED by ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Lustiges und Merkwürdiges von Zahlen und Formen. By Walther Lietzmann. Breslau, F. Hirt, 1930. vi+307 pages, price 9.50 RM.

Mathematik und Bildende Kunst. By Walther Lietzmann. Breslau, F. Hirt, 1931. 150 pages.

It is safe to say that any book written by Dr. Lietzmann is worth reading for three good and sufficient reasons: (1) He knows mathematics from having sat at the feet of Klein as a student and having been closely associated with him in the International Commission on the Teaching of Mathematics from 1908

until the war began; (2) he knows the teaching problem, having tried out the Klein ideas and being at present an Oberstudiendirektor in Göttingen; (3) he writes in a style that appeals to teachers and pupils alike. Such a combination is rare, and hence such books as he writes are unusual.

The first of the works under review is well known to those who follow mathematical courses of the secondary schools of Germany. It first appeared in 1922 and was revised in 1928 and 1929, the present edition being the fourth and containing much material not found in the first two. The work is divided into three parts, which are also bound separately for those who prefer this form. The first covers the general field, with amusing problems, verses, dramatizations (with interesting excerpts from American sources), aphorisms, illustrations, games and fallacies. The second is devoted to numbers, with a collection of some of the best of the number puzzles of the past and some very good ones of recent years. Teachers wishing a supply of amusing problems in this field could not do better than consult this part of the book. Among the problems is this curious one (p. 173): "The American mathematician Augustus de Morgan said, when asked his age, 'I was x years old in the year x^2.' Knowing that he lived in the 19th century, when was he born?" Following out the lighter vein of the book, it might be well to add that the place in which he was born in America can be found by adding 1 to the sum of the two complex cube roots of this number to be added, and to the result adding the five fifth roots of unity. In the same lighter vein it may be stated that he always wrote his name with a capital letter used for that one which, in the Latin alphabet, corresponds to the one in the acrophonic Greek numerals, which represents the number ten.

The second of the books under review is a more recent study by Dr. Lietzmann, and an interesting one. It is largely a collection of pictures in which geometry plays an important part. It runs the course from the pyramids and the rope-stretchers of Egypt, through the perspective of the Renaissance, to the chamber of horrors of the modern jazz of interior decoration. On the way, however, some of the world's best art is shown, chiefly for the purpose of illustrating symmetry and proportion and of illustrating the familiar "golden section" in architecture. Even for those teachers of geometry who do not read German these pictures will be of service, illustrating as they do a quotation of a celebrated architect quoted in the preface, "Ars sine scientia nihil est,"—art without science is nothing. Personally, this reviewer does not believe it, but he is thoroughly in sympathy with this revision,—"Scientia sine arte nihil est." Dr. Lietzmann's notable success illustrates this very point. He teaches with the soul of an artist. Personally, too, this reviewer would have made from the world's architecture a somewhat different selection, as also from the world's gems of the potter's art, and the goldsmith's, and the etcher's, and that of the craftsmen of the Orient. But such criticism is egotistical,—chacun à son gout,—and Dr. Lietzmann's taste is one to be praised as helpful to the teachers of our gild.

DAVID EUGENE SMITH

Moderne Algebra, Teil I. B. L. v. d. Waerden: Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Vol. 33. Julius Springer, Berlin, 1930. 243+8 pages.

The domain of abstract algebra has at present a flourishing period in Germany, and a number of recent textbooks on the subject gives evidence of this development. The present book by v. d. Waerden gives a very clear and satisfactory treatment of the main ideas of the theory of groups, fields and some of the fundamentals of the theory of rings and ideals. It shows many points of connection with the recent book by O. Haupt. The second volume promises to be of still higher interest; it will, according to the plans of the author, deal with general ideal theory and elimination, linear algebras and hypercomplex systems, and the so-called theory of representation. A more complete review of this book will be printed in the *Bulletin*.

OYSTEIN ORE

Darstellende Geometrie, I and II. By Dr. Robert Haussner. Sämmlung Göschen, Nos. 142 and 143.

These numbers of the Göschen collection are revisions and extensions of earlier editions by the same author. The introductory number, which is the fourth edition of the text, deals with the general principles of parallel and orthogonal projection with applications to plane and space figures. The second part, a third edition, treats of harmonic properties and cross-ratios with applications to conic sections. Pascal's and Brianchon's theorems, the pole and polar theory, and many metric properties, including those of the foci, are deduced. The treatment is not essentially different from that found in texts on the geometry of position. Many diagrams appear in each of the texts. These are well drawn and are remarkably clear when the restricted size of the page is considered. In each number also there is a helpful table of the literature of descriptive geometry.

THOMAS F. HOLGATE

Addition-Subtraction Logarithms, to Five Decimal Places. By L. M. Berkeley. White Book and Supply Company, 36 West 91st Street, New York, 1930. xii+136 pages, paper bound.

Addition-subtraction logarithms are based on the following simple relations:

$$\log (a + b) = \log \frac{a + b}{a} + \log a = \log \left(1 + \frac{1}{n}\right) + \log a$$

$$\log (a - b) = \log a - \log \frac{a}{a - b} = \log a - \log \left(\frac{n}{n - 1}\right),$$

where $n = a/b$.

Further, if $m = n+1$, then $1+(1/n) = m/(m-1)$. The present table consists essentially of three parallel columns, of $\log n$, $\log [1+(1/n)]$, $\log (n+1)$, respectively. To find $\log (a+b)$, we find $\log n = (a/b) = \log a - \log b$, then enter the table with $\log n$, take out $\log [1+(1/n)]$, then add this to $\log a$, in accordance

with the first of the above equations. A similar procedure yields $\log(a-b)$. Since in these tables no interpolation is necessary, the result can no doubt be obtained more easily than by the usual process of finding and adding the numbers themselves.

The tables are excellently arranged and clearly printed. A more substantial binding would be necessary if they were to be used extensively.

R. A. J.

Non-Interpolating Logarithms, Cologarithms, and Antilogarithms. By Frederick W. Johnson. The Simplified Series Publishing Company, 1381 Third Avenue San Francisco.

By a simple and ingenious device of arrangement, this table makes it possible to find at once and without interpolation the five-figure mantissa of the logarithm of any five-digit number. The table of logarithms occupies eighteen double pages, and there is also a table of antilogarithms of the same size. The page has an objectionably unfamiliar look, but the reviewer finds the table very workable in practice. One would like to see it tried out in competition with the "Graphic Table" issued a few years ago.

Following the five-place tables is a similarly arranged set of four-place tables. The antilogarithmic tables of the latter are also printed separately in a pamphlet offered by the publishers for free distribution.

R. A. J.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3487. *Proposed by Vladimir F. Ivanoff, San Francisco, Calif.*

Prove the identities:

$$\frac{d^2y}{dx^2}+\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3\equiv 0, \tag{1}$$

$$\frac{d^3y}{dx^3}\left(\frac{dx}{dy}\right)^2+\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^2+3\frac{d^2y}{dx^2}\,\frac{d^2x}{dy^2}\equiv 0, \tag{2}$$

the derivatives d^ny/dx^n and d^nx/dy^n being taken of the same function, $f(x, y)=0$.

3488. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

Find numbers m such that the square of each of the $\phi(m)$ numbers less than m and prime to it will be congruent to 1, modulo m.

UNSOLVED PROBLEMS

Solutions are desired for the following problems, which are the remaining unsolved problems proposed in 1915.

309 [1915, 162]. *Proposed by Jos. B. Reynolds.*

The tangent at one cusp of a vertical, three-arched hypocycloid is horizontal, and a particle will just slide under gravity from the upper cusp to this cusp. Find the equation which the coefficient of friction must satisfy.

232 [1915, 202]. *Proposed by E. T. Bell.*

If $F(x)$ is any function of x which vanishes with x, and which, for $0 < |x| \leqq |\xi|$, can be expanded in an absolutely convergent series of positive powers of x, show that a function $f(n)$ may be found, essentially in one way only, such that

$$\int_0^{\xi} \frac{1}{x} F(x)dx = -\log \prod_{n=1}^{\infty} (1 - \xi^n)^{(1/n)f(n)},$$

and find the form of $f(n)$ explicitly in terms of the coefficients in the expansion of $F(x)$. Hence, as particular examples, expand (when possible by the method) e^z as an infinite product, and show that

$$\frac{1}{e} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{2^n}\right)^{(1/n)\phi(n)},$$

where $\phi(n)$ is the totient of n.

313 [1915, 202]. *Proposed by Clifford N. Mills.*

A heavy extensible wire of length c and of constant cross-section w, and density k, is suspended by one end and hangs vertically. If e is the coefficient of elasticity, show that the length of the wire when stretched will be $c(1+ekgw/2)$.

234 [1915, 268]. *Proposed by Frank Irwin.*

Start with any number, for instance 89, and add to it the number obtained by reversing the order of its digits: $89+98=187$. Now perform the same operation on the result: $187+781=968$. If we continue in this way we arrive, after a certain number of operations, at a number which reads the same forward and backwards (24 operations bring us to 8813200023188). Will this be the case no matter with what number we start?

NOTE: I am told that this is an old problem, but do not know whether it has ever been solved. (No other number under 100, except, of course, 98, requires so many operations to lead to the desired result as 89 does.)

308 [1915, 268]. *Proposed by H. S. Uhler.*

Prove that when a ray of light passes obliquely through a prism in such a manner as to maintain a constant value for the total deviation of the projection of the ray on the principal section, the ray inside the prism generates a cone of elliptical right section, It is assumed that the prism is surrounded by a medium having a smaller index of refraction than the index of the material of the prism.

315 [1915, 309]. *Proposed by H. S. Uhler.*

A solid, homogeneous, right, circular cylinder is allowed to move from rest down a circular cylindrical track which is concave upwards. Find the ratio of the radius of the track to the radius of the cylinder when the time of descent through a finite arc to the bottom is the same for the extreme cases of no slipping and zero friction. Show also that the same relation holds for a sphere descending a cylindrical or spherical surface.

SOLUTIONS

3441 [1930, 380]. *Proposed by Solomon Kullback, Brooklyn, New York.*

In a given triangle, inscribe an equilateral triangle having a given point, P, on one of its sides. Suggested by problem No. 3405.

Solution by Ralph Deutsch, Brooklyn, N. Y.

Construction: Draw the arc of a circle AOB so that it contains the angle $C+60°$, and the arc BOC so that it contains $A+60°$. Let O be the intersection of these two arcs, and let the angles that the lines OA, OB, OC make with the sides of ABC be numbered as in the figure. Construct the angle OYP equal to angle 1 with Y lying upon CA, and then the angle OYX equal to angle 5, where YX cuts BC in X. Extend YP until it cuts AB in Z. Then XYZ is the required triangle.

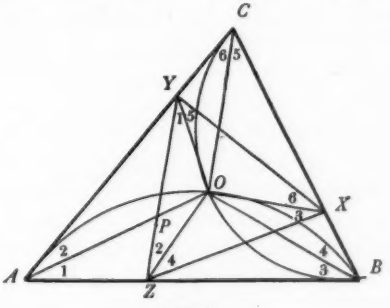

FIG. A.

Proof: Since O, Y, A, Z and O, Y, C, X are concyclic, O, X, B, Z are also concyclic. Therefore, angles OZY, OXY, OXZ, and OZX are equal, respectively, to angles 2, 6, 3, and 4. Also $\angle AOB+\angle BOC+B+1+5=360°$, or $C+A+B+1+5=240°$. Hence $1+5=60°$. Since $\angle AOC=60°+B$ it can be proved in a like manner that $2+4=60°$. This shows that XYZ is equilateral.

A discussion by Otto Dunkel: In the construction above the points X, Y, Z

have the same sense of rotation as A, B, C, and this case will be considered first. In general, if P lies within a certain region no construction will be possible, and for certain other regions the construction above is impossible although there may exist solutions. The directions for the construction and also the number of solutions vary with the location of P. There are four types of regions,and in order to locate these regions the requirement that a side of the required triangle shall pass through a given point P will be dropped.

The three envelopes. If no angle of the triangle ABC is as great as 120° the point O lies within this triangle, and this case will be considered first. For convenience suppose that XYZ is an equilateral triangle with X lying within the segment BC, Y lying within CA, and Z lying within AB. Then the three circles through A, Y, Z:B, Z, X:C, X, Y meet in a point, which is easily shown to be O; and all the equalities between the angles of the figure given above easily follow. Also $\angle YOZ = 180° - A$, $\angle ZOX = 180° - B$, $\angle XOY = 180° - C$. It will be shown later how to construct one triangle $X'Y'Z'$ which is always inscribed in the strict sense above. When the angle YOZ rotates about O remaining always $180° - A$, the points Y and Z trace projective ranges on CA and on AB, respectively. Hence the envelope of YZ is a conic tangent to these last two lines. When Z is at A, Y is at the point of tangency B_a of CA, where $\angle OB_aA = \angle OAB$. Similarly, the point of tangency of AB is C_a, where $\angle OC_aA = \angle OAC$. When OY is parallel to AC, OZ is parallel to AB; and hence YZ is now the line at infinity. The conic is therefore a parabola. It also follows that O is the focus, for it is known that the circle circumscribing the triangle formed by three tangents to a parabola always passes through its focus. Here such a triangle is AYZ, and its circumscribed circle always passes through O. Hence YZ, ZX, XY are tangent always to corresponding parabolas α, β, γ with the common focus O. Moreover, from the theory of the parabola we know that the angles of the triangles OYZ, OXY, OZX remain constant. Hence if the figure formed by OX, OY, OZ rotates about O as a rigid figure preserving the angles at O, the lines XY, YZ, ZX will be tangent to their respective parabolas and will always form an equilateral triangle, since they do so for an initial position. Hence as Y traces the entire line on which CA lies we obtain all the inscribed equilateral triangles of the same order of vertices as ABC. These triangles contain O, and therefore a part or the whole of the area of ABC. Let X', Y', Z' be the feet of the perpendiculars from O to BC, CA, AB; then on account of the values of the angles subtended by pairs of these points at O, $X'Y'Z'$ is an equilateral triangle. A simple consideration of the figure will show that it is inscribed in the above strict sense. Let the perpendicular from O to $Y'Z'$ cut it in V_a and CA in B_a'. Then, since $\angle OY'Z' = \angle OAB = \angle OB_aA = \angle OB_a'B_a$, the triangle OB_aB_a' is isosceles with vertical angle at O. Hence OV_a is the axis of α and V_a is its vertex. Thus the feet of the perpendiculars V_a, V_b, V_c from O to the sides of $X'Y'Z'$ are the vertices of α, β, γ, and the sides of this triangle are the tangents at the vertices.

The regions. The directrices of α, β, γ form an equilateral triangle with sides

parallel to those of $X'Y'Z'$, with O as center of similitude for the two triangles, and with the ratio of 2:1 for the corresponding parts. The intersections of any pair of the parabolas must lie upon an altitude of the directrix triangle, and hence the regions have in general six angular points lying in pairs upon the three altitudes of this directrix triangle. The focus O may lie anywhere within $X'Y'Z'$ according to the shape of ABC, but not on a side if we have an actual triangle ABC or a triangle in this case in which no angle is as great as 120°. Let I' and r' be the center and radius of the circle (I') inscribed in $X'Y'Z'$, while I'' and r'' are the corresponding symbols for the directrix triangle. If O lies within (I'), then $OI''=2\,OI'<2r'=r''$. Since for this position of O, $OI''<r''$, the point I'' lies within each parabola and therefore in the same region as O, which will be denoted by I. In this case α, β, γ intersect in three pairs of points, the points of each pair lying upon an altitude of the directrix triangle and separated by its center I''. The figure shows this case. There are in addition to the region I three regions II_a, II_b, II_c such that from any point within II_a, for example, two tangents may be drawn to α but none to β or γ; three regions III_{ab}, III_{bc}, III_{ca} such that from any point within III_{ab}, for example, two tangents may be drawn to α and two to β but none to γ; three regions IV_a, IV_b, IV_c such that from a point within any one of them two tangents may be drawn to each parabola.

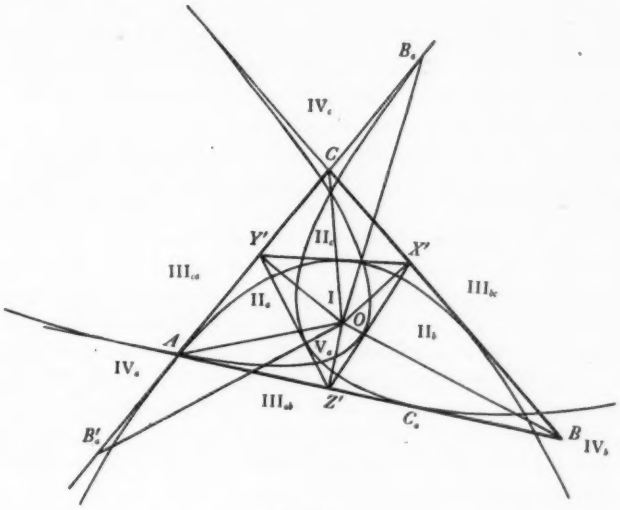

FIG. B.

If O lies outside the circle (I') but within $X'Y'Z'$, $OI''>r''$ and here I'' lies outside of each parabola. In this case the two points of each pair of intersections lie on an altitude on the same side of I'', and there are only two regions II but four regions III. If O lies on (I') but not at a point of tangency, $OI''=r''$

and all three parabolas pass through I''. In this case a region II has reduced to a point and there are only nine regions.

Constructions. If P lies within I there is no construction; if within a II there are two; if within a III, four; if within a IV, six. If P lies on the boundary of any of these regions there may be one, two, etc., up to five constructions depending upon the particular boundary. The cases are easily distinguished. If P lies for example within III_{bc}, an arc of a circle is described on OP as a chord so that it contains the angle OCB and lies on the side of OP towards C. This arc cuts AC in two points each of which serves for the Y vertex, and the balance of the construction easily follows. The lines PY are tangents to γ. Then an arc is drawn on OP on the side towards B so as to contain the angle OBC. This arc cuts AB in two points each of which serves for a Z. The lines PZ are tangent to β; again the balance of the construction follows in a simple manner. In this way four equilateral triangles are constructed for this position of P; and for each region definite directions may be given in a similar manner for the constructions.

The remaining cases. In the above no angle is as great as 120°. If an angle, say C, is 120°, then O lies at Z', and every triangle XYZ has a vertex at $O \equiv Z'$. Also CO bisects the angle C; the two parabolas α and β reduce to the point O. If P lies outside γ there are four constructions; if inside, but not at O, there are two; if at O, there are an infinite number.

If an angle, say C, is greater than 120°, O lies outside ABC and hence also outside $X'Y'Z'$, but within the vertical angle of 60° at Z'. This case may be treated in a manner similar to the above case for $C<120°$, but there is a slight difference in the expressions for the equalities of the angles. Here $\angle AOC = 60° + B$, $\angle COB = 60° + A$, $\angle AOB = 300° - C$. The parabolas α and β intersect on the internal bisector of the angle of the directrix triangle corresponding to Z', while the other two lines of intersections are the external bisectors of the angles corresponding to X' and Y'. The interiors of α and β lie outside ABC. The intersection of the three lines above is the center of an escribed circle of the directrix triangle, and it lies always outside the three parabolas. Thus the ten regions contain, as in the second case above, only two regions II, II_a and II_b; and of the four III regions there are two separate regions III_{ab}. The region I is necessarily outside ABC.

If we consider the triangle XYZ with an order of rotation opposite to that of ABC, there will be still other constructions possible. This gives rise to a second point O'. If $C>60°$, $B>60°$, then $\angle AO'B = C - 60°$, $\angle CO'A = B - 60°$, $\angle BO'C = 60° - A$. If $A<60°$, $B<60°$, then $\angle AO'B = C - 60°$, $\angle CO'A = 60° - B$, $\angle BO'C = 60° - A$. These constructions include the cases in which XYZ lies outside ABC, and they include, of course, no cases of a triangle inscribed in the strict sense. Let J' be the center of the escribed circle (J') in the angle of $X'Y'Z'$ in which O' lies, then as before there will be three cases according as O' lies inside, on, or outside (J'). The treatment of this case is similar to the above, and there may be from one to six, or no constructions according to the region

in which P lies. This concludes the cases arising from the various locations of the focus with respect to $X'Y'Z'$.

The problem of inscribed triangles XYZ of any given form may be treated in a similar manner, but there would be a greater variety of cases.

The minimum triangle. It is worthy of note that $X'Y'Z'$ is the inscribed equilateral triangle of minimum perimeter and minimum area. For in all cases

$$YZ = \frac{OY \sin A}{\sin OAC};$$

and, since YZ varies directly as OY, the minimum of YZ occurs when OY is perpendicular to CA.

3442 [1930, 380]. *Proposed by the late F. P. Matz.*

Solve the equation

$$\frac{d}{dw}\left(\frac{dw}{dr} + \frac{2w}{r}\right) = 0.$$

Solution by Eugene A. Rasor, Ohio State University

A first integration gives

$$\frac{dw}{dr} + \frac{2w}{r} = -\frac{3}{a}.$$

After multiplying by the integrating factor r^2, a second integration gives the final result,

$$r^3 + ar^2w + b = 0.$$

Also solved by A. W. Bailey, E. C. Kennedy, J. D. Leith, and A. Pelletier.

3445 [1930, 381]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

If p is odd and greater than 1, prove that

(a) $$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \bmod p;$$

(b) $$2^2 \cdot 4^2 \cdot 6^2 \cdots (p-1)^2 \equiv (-1)^{(p+1)/2} \bmod p.$$

Solution by W. Randolph Church, Hightstown, N. J.

The theorem of Wilson states that

(1) $$1 \cdot 2 \cdot 3 \cdot 4 \cdots (p-1) \equiv -1 \pmod p,$$

where p is any odd prime. We have the following set of congruences:

(2) $$i \equiv -(p-i) \pmod p, \quad p = 0, \pm 1, \pm 2, \cdots.$$

Replace the $(p-1)/2$ even numbers in the left hand side of (1) by the numbers

congruent to them as obtained by letting $i=2, 4, 6, \cdots, (p-1)$ in (2). After combining we have

$$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \ (-1)^{(p-1)/2} \equiv -1 \pmod{p},$$

or

(3)
$$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Similarly, replace the $(p-1)/2$ odd numbers in (1) by the numbers congruent to them as obtained by letting $i=1, 3, 5, \cdots, (p-2)$ in (2). This gives

$$2^2 \cdot 4^2 \cdot 6^2 \cdots (p-1)^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Also solved by L. S. Kennison, L. C. Mathewson, and the Proposer.

3448 [1930, 446]. *Proposed by Oliver D. Kellogg, Harvard University.*

Prove that whenever the infinite series, with positive terms,

$$u_1 + u_2 + u_3 + \cdots$$

converges, the series

$$\frac{u_1}{r_1} + \frac{u_2}{r_2} + \frac{u_3}{r_3} + \cdots$$

diverges,

$$r_n = u_n + u_{n+1} + u_{n+2} + \cdots$$

being the remainder of the first series after $n-1$ terms.

Solution by the Proposer.

By definition, $r_n = u_n + r_{n+1}$. Hence

$$\frac{r_{n+1}}{r_n} = 1 - \frac{u_n}{r_n},$$

$$\frac{r_{n+2}}{r_n} = \left(1 - \frac{u_n}{r_n}\right)\left(1 - \frac{u_{n+1}}{r_{n+1}}\right) > 1 - \frac{u_n}{r_n} - \frac{u_{n+1}}{r_{n+1}},$$

and, as is easily established by induction,

$$\frac{r_{n+p}}{r_n} > 1 - \frac{u_n}{r_n} - \frac{u_{n+1}}{r_{n+1}} - \cdots - \frac{u_{n+p}}{r_{n+p}}.$$

As the given series converges, r_{n+p} approaches 0 as p becomes infinite. Hence, to any n, there corresponds a p, such that

$$\frac{u_n}{r_n} + \frac{u_{n+1}}{r_{n+1}} + \cdots + \frac{u_{n+p}}{r_{n+p}} > 1 - \frac{r_{n+p}}{r_n}$$

exceeds any preassigned number less than 1. It follows that the series

$$\frac{u_1}{r_1}+\frac{u_2}{r_2}+\frac{u_3}{r_3}+\cdots$$

diverges, as was to be proved.

Also solved by Ralph Agnew, C. H. Dix, R. L. Jeffrey, J. D. Leith, J. F. Locke, G. E. Raynor, F. Underwood, and Morgan Ward.

NOTE: Professor Raynor, in his solution, calls attention to the fact that this theorem is a special case of Dini's theorem a proof of which is given on page 293 English translation of Knopp's *Theory and Application of Infinite Series.* Professor Norman Miller also cites this authority.

The Proposer notes that it seems amusing that exactly this result should be needed in the study of a problem in potential theory where apparently he came across it.

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus Ohio.

THE SUMMER SESSION FOR TEACHERS OF MATHEMATICS TO BE HELD BY THE SOCIETY FOR PROMOTION OF ENGINEERING EDUCATION

One of the sessions of the Summer School for Engineering Teachers, held annually by the Society for the Promotion of Engineering Education, will be devoted this year to the study of methods of teaching mathematics.

This Summer School, a unique undertaking in the field of higher education, was established in 1927 as one of the outgrowths of a general investigation of Engineer Education begun by the S.P.E.E. in 1924. The school has as its purpose intensive study and discussion of methods of teaching the principal subjects or divisions of engineering education. Sessions are held annually in various institutions throughout the country and are devoted to particular portions of the engineering curriculum. The Session of 1927, with which the school was inaugurated, dealt with the teaching of Engineering Mechanics. Sessions were held at Cornell and the University of Wisconsin. Since that year other Sessions of the school have been devoted to the teaching of Physics, Electrical Engineering, Mechanical Engineering, Engineering Drawing and Descriptive Geometry, and Civil Engineering. These sessions have been held at the Massachusetts Institute of Technology, The University of Pittsburgh, Purdue University, the Carnegie Institute of Technology, and Yale University.

From modest beginnings the school has developed into one of the really important enterprises in the field of engineering education. Sessions are attended by teachers from all parts of the country, the enrolment ranging from 50 to 100 individuals. The teaching staffs are recruited from among the foremost teachers, scientists, and practising engineers.

It will be of interest to teachers of mathematics to learn that the Summer School Session of this year will be held at the University of Minnesota immediately preceding the annual meetings of the Mathematical Association of America and the American Mathematical Society. The S.P.E.E. session will open on August 24th and continue for two weeks, closing on September 5th. The program of the Session is arranged in five principal divisions:

1. General principles of teaching.
2. Teaching of mathematics to engineering students.
3. Advanced mathematics.
4. Application of mathematical principles in engineering practice.
5. Historical and miscellaneous.

These five divisions of the program are carried forward simultaneously by means of sequences of lectures and discussion periods, informal conferences and the like. As an illustration, lecture and conference periods on the following topics are scheduled in the second division of the program:

Teaching the Fundamentals of Trigonometry.

The Teaching of College Algebra (A) Arrangement of course content.
(B) Teaching problems.

Analytic Geometry (A) Arrangement of course problems.
(B) Teaching problems.

Differential Calculus (A) Arrangement of course content.
(B) Teaching the nature of the derivative.

Integral Calculus (A) Arrangement of course content.
(B) Teaching problems.

Differential Equations for Engineering Students.

Among the topics in other divisions of the program is a series of lectures and discussions on the general principles of teaching. These are to be conducted by a teacher of educational method. A part of the program will be devoted to specific educational problems, such as the question of the size of class sections, the coordination of mathematics with the related engineering subjects, the adjustment between college and secondary school mathematics, and combined versus unit courses in mathematics. Two lectures on the history of mathematics are scheduled, as well as a number of lectures on higher mathematics and on relationship between mathematical principles and current engineering practice.

The teaching staff of the Session is headed by Dean O. M. Leland, College of Engineering and Architecture, University of Minnesota, who will serve as the Director of the Session, and by Professor C. A. Herrick of the University of Minnesota who will act as the Secretary. Other members of the staff include—F. T. Spaulding, of the Graduate School of Education, Harvard University; E. R. Hedrick, University of California at Los Angeles; Louis O'Shaughnessy, Virginia Polytechnic Institute; S. P. Timoshenko, University of Michigan; Dunham Jackson, W. E. Brooke, and M. E. Haggerty, University of Minnesota; H. L. Rietz, State University of Iowa; W. J. Berry, Polytechnic Institute of Brooklyn; R. C. Archibald, Brown University; E. V. Huntington, Harvard University; J. W. Young, Dartmouth College; Warren Weaver and C. S. Slichter, University of Wisconsin; Thornton C. Fry, Bell Telephone Laboratories; Charles N. Moore, University of Cincinnati; Leigh Page, Yale University.

Attendance at the Session is open to teachers of mathematics in general and particularly to teachers of mathematics to engineering students. The cost of attendance, aside from traveling expenses, is limited to a registration fee of $10 and to the charge for room and meals which will be approximately $35 for the two week period.

H. P. Hammond of the Polytechnic Institute of Brooklyn is the general director of the S.P.E.E. Summer Schools. Further information concerning the session can be secured from Professor Hammond by addressing him at 99 Livingston Street, Brooklyn, New York.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Dr. N. B. MacLean, formerly professor of mathematics at the University of Winnipeg, has been appointed professor of mathematics at McGill University, as successor to Professor D. A. Murray, retired.

The following appointments to instructorships in mathematics are announced:

Iowa State College, J. C. Hempstead.

University of Michigan, Dr. J. D. Elder.

Dr. R. A. Fisher, Chief Statistician of Rothamsted Experimental Station, Harpenden, England, will be on the staff of the Department of Mathematics at the University of Minnesota during the second half of the summer session, July 27 to August 29. He will lecture on "The Theory of Estimation" and on "The Statistical Theory of Experimental Design". Professor Griffith C. Evans of the Rice Institute will also be in residence at the University of Minnesota during the entire summer session. During the first term, from June 17 to July 25, Professor Evans will lecture on "The Mathematical Theory of Economics". In the second term he will present "Potential Theory". The courses of Professor Evans and Dr. Fisher are in addition to the usual summer offerings of the Department of Mathematics.

The following courses in mathematics are announced for the summer 1931:

University of Chicago, first term, June 22–July 24; second term, July 27–Aug. 28. In addition to the regular courses in differential and integral calculus, the following advanced courses will be offered: By Professor H. E. Slaught: Differential equations; Elliptic integrals. By Professor G. A. Bliss: Algebraic functions. By Professor A. C. Lunn: Thermodynamics; Vector analysis in Riemann-Einstein space. By Professor M. I. Logsdon: Projective geometry of Hyperspaces. By Professor L. M. Graves: Advanced calculus; Theory of functions of real variables. By Professor R. W. Barnard: Linear algebras. By Professor E. Hille: Theory of equations; Fourier series. By Professor O. Zariski: Synthetic projective geometry; Applications of linear series on an algebraic curve. By Professor W. Bartky: Celestial mechanics; Modern theories of differential equations I.

University of Colorado, first term, June 22 to July 25; second term, July 27 to August 28. In addition to the usual elementary work in algebra, trigonometry, analytic geometry, and calculus, the following courses will be offered: First term —By Professor Light: Teachers' course in mathematics; Statistics; Differential equations of mathematical physics. By Professor Kempner: Differential equations; Functions of a complex variable. Second term—By Professor Light: His-

tory of mathematics; Theory of finance; Differential equations of mathematical physics (continued). By Professor Kempner: Differential equations (continued); Functions of a Complex variable (continued); Theory of equations.

University of Michigan, June 29 to August 21. In addition to courses in algebra, trigonometry, analytic geometry, elementary calculus, statistics, and finance, the following advanced courses will be offered: By Professor J. W. Bradshaw: Projective geometry; Teaching of geometry. By Professor P. Field: Vector analysis; Engineering problems. By Professor W. B. Ford: Advanced calculus; Infinite series. By Professor T. H. Hildebrandt: Topics in calculus; Functions of a real variable. By Professor G. Y. Rainich: Mathematics of relativity. By Professor L. A. Hopkins: Analytic mechanics. By Professor H. C. Carver: Mathematical theory of statistics. By Professor R. L. Wilder: Foundations of mathematics; Theory of sets of points. By Professor Norman H. Anning: Differential equations; History of mathematics. By Professor W. L. Ayres: Foundations of Euclidean geometry. By Professor C. J. Coe: Integral equations. By Professor A. H. Copeland: Theory of probability. By Professor N. C. Fisk: Graphical methods. By Professor J. A. Nyswander: Finite differences.

Northwestern University, June 22 to August 15. In addition to courses in trigonometry, college algebra, analytic geometry, differential and integral calculus, the following advanced courses will be offered: By Professor E. J. Moulton: Theory of numbers. By Professor F. E. Wood: Teaching of mathematics in secondary schools. By Professor H. S. Wall: Advanced calculus.

Ohio State University, June 15 to August 28. In addition to the usual elementary courses in algebra, analytic geometry, and calculus, the following advanced courses will be offered: By Professor H. W. Kuhn: Advanced calculus; Theory of finite groups. By Professor H. Blumberg: Introduction to the theory of relativity; Methods and problems in the theory of real functions. By Professor F. R. Bamforth: Solid analytic geometry; Modern theories in ordinary differential equations. By Dr. P. M. Swingle: Introduction to higher algebra.

University of Pennsylvania, July 6 to August 15. In addition to elementary courses, the following advanced courses will be offered: By Professor H. H. Mitchell; Projective geometry. By Professor G. G. Chambers: Advanced calculus. By Professor S. P. Shugert: Galois theory of equations. By Professor P. A. Caris: Diophantine analysis. By Professor J. A. Shohat: Theory of approximation and differential equations.

University of Pittsburgh, June 15 to August 21. In addition to the usual undergraduate courses, the following advanced courses will be offered. By Professor K. D. Swartzel: Functions of a complex variable; Teaching of mathematics. By Professor F. A. Foraker: Modern synthetic geometry; Solid analytic geometry. By Associate Professor Taylor: Advanced calculus; Functions of a real variable. By Assistant Professor Culver: Differential equations; Theory of equations.

University of Texas, first term, June 9 to July 20; second term, July 18 to August 31. In addition to freshman courses given in both terms, the following courses will be offered. First term—By Professor R. L. Moore: Foundations of geometry; Functions of real variables. By Professor E. L. Dodd: Probability; Infinite processes. By Professor H. J. Ettlinger: Ruler and compass constructions; Research in differential equations. By Professor H. S. Vandiver: Number theory. By Professor P. M. Batchelder: Teaching problems in mathematics. By Professor Mary E. Decherd: Calculus. By Professors C. M. Cleveland and H. V. Craig: Advanced calculus. Second term—By Professor R. G. Lubben: Non-Euclidean geometry; Analytic functions. By Professor J. H. Roberts: Calculus; Functions of real variables. By Professor E. G. Keller: Advanced calculus; Potential theory.

University of Wisconsin, six weeks session, June 29 to August 7. By Professor H. P. Evans: Differential equations; Theory of equations; Definite integrals. By Professor W. W. Hart: Advanced course in the teaching of high school mathematics; Teaching of mathematics. By Professor R. E. Langer: Advanced calculus; Modern analytic geometry. By Professor M. Marden: Differential and integral calculus. By Professor Warren Weaver: Mechanics of a particle. Special nine weeks session for graduates, June 29 to August 28. By Professor M. H. Ingraham: Introduction to general analysis; Theory of numbers. By Professor Warren Weaver: Tensor analysis and its applications to the theory of relativity.

Information has just been received from Japan of the death of Norifumi Okamoto, who has recently been connected with the Imperial Academy of Tokyo, and has been in charge of the cataloguing of the old mathematical manuscripts contained in that library.

Mr. Okamoto was one of the last representatives of the old Japanese school of mathematics, but in addition to his knowledge of the ancient works, he was thoroughly conversant with the nature of the western mathematics introduced after the Restoration of 1868. In his younger years he translated some of the western literature in this subject. He had a long experience in teaching in the normal schools, and Peers' College, and the Military Officers' School.

He had not completed the work of the cataloguing of the large collection of ancient mathematical books and manuscripts, but it is to be hoped that this can be carried on by some one who has worked with him. It would seem to American scholars that Mr. Mikami would naturally take up such an important work.

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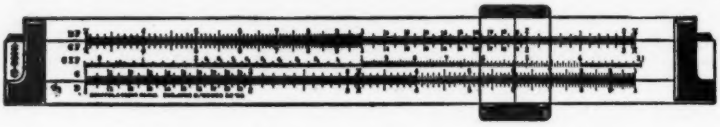

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Cohen, Abraham, An Introduction to the Lie Theory of One-Parameter Groups with application to the solution of Differential Equations (1911) Cloth. Reprint 1931....$2.00

F. Klein's Famous Problems of elementary Geometry, ed. Beman & Smith, rev. by R. C. Archibald. 1930. Cloth. XI+92 pages.....$1.50

Of A. G. Webster "Partial Differential Equations of Mathematical Physics" a new edition is in preparation and will be ready within about a year. A German translation with revision by Dr. G. Szegö of Koenigsberg is now available at $7.00.

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THE CARUS MONOGRAPH COMMITTEE is pleased to announce that the first edition of Number Four is well advanced in sales and that each of the others has gone into a second edition; also that a German Edition of Number One is being brought out by the firm of Teubner in Leipzig and Berlin. The titles of the monographs are: (1) "Calculus of Variations" by Professor GILBERT A. BLISS; (2) "Analytic Functions of a Complex Variable" by Professor DAVID R. CURTISS; (3) "Mathematics of Statistics" by Professor HENRY L. RIETZ; "Projective Geometry," by Professor JOHN W. YOUNG.

The price of these Monographs is $1.25 per copy to institutional and individual members of the Association when ordered directly through the Secretary, one copy to each member; this is the bare cost of production. The price to all non-members of the Association and for all quantity orders for class use is $2.00 per copy, obtained only through the Open Court Publishing Company, 337 East Chicago Avenue, Chicago, Illinois, distributors to the general public of Association publications.

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS, Peoria, May 1-2.
INDIANA, Muncie, May 1-2.
IOWA, Davenport, May 1-2.
KANSAS, Topeka, Jan. 24.
KENTUCKY, April 15.
LOUISIANA-MISSISSIPPI, Natchitoches, La., March 13-14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA, St. John's University, Collegeville, May 16.
MISSOURI, St. Louis, November.
NEBRASKA, Lincoln, May.
OHIO, Columbus, April 2.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Boulder, Colo., April 17-18.
SOUTHEASTERN, Auburn, Ala., April 24-25.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 14.
TEXAS, Fort Worth, Tex., Jan. 31.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus Ohio.

A REPORT OF THE COMMITTEE ON COLLEGE ENTRANCE REQUIREMENTS IN GEOMETRY

Early in 1929 a committee was appointed jointly by the Mathematical Association of America and the National Council of Teachers of Mathematics, to study the feasibility of a proposal that college entrance requirements in geometry should be modified so as to bring about the more extensive introduction of courses including the essentials of plane and solid geometry in a single year's work, in place of the traditional year of plane geometry. The committee begs leave to report as follows:

There is a widespread conviction held by a group of teachers impressive both as to numbers and as to scientific and educational standing, that the proposed rearrangement will be a fundamental improvement in the teaching of geometry. The leaders of this group are teachers who believe in the importance of mathematics as part of a liberal education. Their desire is not to diminish the amount of time given to the teaching of geometry, but to increase the effectiveness of that teaching. Their chief concern is to remedy the situation created by the progressive disappearance of solid geometry from the course of study of the average high school pupil, and to bring some important parts of it into the program of pupils who under present conditions would learn nothing of it at all.

The committee believes that they should be given all practicable support and encouragement in their endeavor to solve the problems involved in the rearrangement, and believes in particular that participation by the College Entrance Examination

[1] Presented to the Trustees of the Association at Cleveland, December 31, 1930, and to the Board of Directors of the National Council of Teachers of Mathematics at Detroit, February 20, 1931. A preliminary announcement of the Committee was published in an earlier number of the Monthly, vol. 36 (1929), pp. 347–348, as well as in the Mathematics Teacher for December, 1929.